Question 9

(i) (a)

Next Instruction	R_1	R_2	R_3
1	3	2	0
2	3	2	0
3	3	2	0
4	3	3	0
5	3	3	1
1	3	3	1
6	3	3	1
STOP	1	3	1

(i)(b)

[[If $R_1 \neq R_2$ when instruction 2 is executed we then add one to both these registers. Therefore they will also be unequal next time we reach instruction 2. Therefore we can ignore instruction 2 after it has been executed once.]]

$f_P^1: \mathbb{N} \to \mathbb{N}$ is defined by $f_P^1(n) = 0$.

$f_P^2: \mathbb{N}^2 \to \mathbb{N}$ is defined by $f_P^2(n, m) = \begin{cases} 0, & \text{if } m = 0 \\ n - m, & \text{if } n \geq m > 0 \\ \text{undefined}, & \text{otherwise} \end{cases}$.

$f_P^3: \mathbb{N}^3 \to \mathbb{N}$ is defined by $f_P^3(n, m, q) = . \begin{cases} q, & \text{if } m = q \\ n - m + q, & \text{if } n \geq m \text{ and } m \neq q \\ \text{undefined}, & \text{otherwise} \end{cases}$

[[Check that when we put q = 0 in f_P^3 we get f_P^2, and when we put m = 0 in f_P^2 we get f_P^1.]]

(ii)

1 S(2)
2 J(1, 2, 6) [[Jump if n = 1]]
3 S(2)
4 J(1, 2, 6) [[Jump if n = 2]]
5 S(3) [[Otherwise]]
6 C(3,1)

Question 10

(i) [[See Unit 2, Problem 1.9 for $\leq$.]]

Let $\chi_{\geq}: \mathbb{N}^2 \to \mathbb{N}$ be defined by $\chi_{\geq}(n, m) = \overline{sg}(m \dot{-} n)$.

If $n < m$ then $m \dot{-} n > 0$ and so $\chi_{\geq}(n, m) = 0$.
If $n \geq m$ then $m \dot{-} n = 0$ and so $\chi_{\geq}(n, m) = 1$.

Since $\chi_{\geq}$ is obtained by substitution from the total primitive recursive functions $\overline{sg}$ and $\dot{-}$ then $\chi_{\geq}$ is also a total primitive recursive function.

Therefore $\chi_{\geq}$ is a characteristic function for the relation $\geq$ and so it is primitive recursive.

(ii) [[See Unit 2, Problem 1.10.]]

Since R and S are primitive recursive relations then their characteristic functions χ_R and χ_S, respectively, are primitive recursive functions.

Let $\chi_T(n_1, n_2, \ldots, n_k) = mult(\chi_R(n_1, n_2, \ldots, n_k), \chi_S(n_1, n_2, \ldots, n_k))$.

If either R or S is not true then $mult(\chi_R(n_1, n_2, \ldots, n_k), \chi_S(n_1, n_2, \ldots, n_k)) = 0$ and $\chi_T(n_1, n_2, \ldots, n_k) = 0$.
If both R and S are true then $mult(\chi_R(n_1, n_2, \ldots, n_k), \chi_S(n_1, n_2, \ldots, n_k)) = 1$ and $\chi_T(n_1, n_2, \ldots, n_k) = 1$.

Then the relation T is primitive recursive since its characteristic function is obtained by substitution from the primitive recursive functions mult, χ_R and χ_S .

(iii) Unit 2 Theorem 1.5 will be used

Define the functions

$g_1(n, m) = m + n = add(n, m)$
$g_2(n, m) = 2^n = exp(2, n)$,
$g_3(n, m) = 12 = C_{12}^2(n, m)$,

and the relations

$R_1(n, m) \Leftrightarrow \chi_E(3nm + 1)$,
$R_2(n, m) \Leftrightarrow n + 5m = 9999$,
$R_3(n, m) \Leftrightarrow$ not $R_1(n, m)$ and not $R_2(n, m)$.

Then we can write

$$f(n, m) = \begin{cases} g_1(n, m) & \text{if } R_1(n, m) \\ g_2(n, m) & \text{if } R_2(n, m) \\ g_3(n, m) & \text{if } R_3(n, m) \end{cases}$$

As g_1, g_2, and g_3 can be written the primitive recursive functions C^2_{12} , add, and exp using constants then g_1, g_2, and g_3 are primitive recursive functions.

The characteristic function of the relation R_1, $\chi_{R1}(n, m) = \chi_E(3mn + 1)$. As χ_{R_1} is obtained by substitution from the primitive recursive functions χ_E, mult, and add using constants, then it is a primitive recursive function. Hence R_1 is a primitive recursive relation.

The characteristic function of the relation R_2, $\chi_{R2}(n, m) = \chi_{eq}(n + 5m, 9999)$. As χ_{R_2} is obtained by substitution from the primitive recursive functions χ_{eq}, mult and add using constants, then it is a primitive recursive function. Hence R_2 is a primitive recursive relation.

Using the results of Unit 2, Problem 1.10, then R_3 is also a primitive recursive relation.

From the definition of R_3 it follows that the set of relations R_1, R_2, and R_3 are exhaustive.

If the relation R_1 holds then both n and m are odd. If the relation R_2 holds then both n and m cannot be odd. Therefore R_1 and R_2 are mutually exclusive. From the definition of R_3, if the relation R_3 holds then neither R_1 or R_2 holds. Therefore R_1, R_2 and R_3 are mutually exclusive and exhaustive.

Since all the conditions required for the use of Theorem 1.5 of Unit 2 hold then it follows that f is primitive recursive.

Question 11

(i)(a)

g(n_1, n_2, n_3) = $f\left(U_3^3(n_1, n_2, n_3), \text{succ}\left(U_1^3(n_1, n_2, n_3)\right)\right)$.

As g is obtained by substitution from the primitive recursive function f and the basic primitive recursive functions succ , U_1^3 , and U_3^3 , then g is a primitive recursive function.

(i)(b)

Let fac(0) = 1
and fac(m + 1) = g(m, fac(m))
where g(n_1, n_2) = $mult\left(\text{succ}\left(U_1^2(n_1, n_2)\right), U_2^2(n_1, n_2)\right)$.

g is a primitive recursive function since it is obtained by substitution from mult and the basic primitive recursive functions succ , U_1^2 , and U_2^2 . As fac is formed by primitive recursion from the constant 1 and the primitive recursive function g, then fac is a primitive recursive function.

(ii)

$f(n) = \mu y(n < \text{fac}(y))$
Consider the relation R given by
$R(n, y) \Leftrightarrow n < \text{fac}(y)$.
The characteristic function of the relation R is $\chi_R(n, y) = \chi_<(n, \text{fac}(y))$.
which is obtained by substitution from the primitive recursive functions $\chi_<$ and fac and so is primitive recursive.

By Unit 2, Theorem 3.5 the function $g : \mathbb{N}^2 \rightarrow \mathbb{N}$ given by
$$g(n, z) = \mu y \leq z\ R(n, y)$$
is primitive recursive.

If the bound for z is chosen as n + 1 then we have
f(n) = g(n, n+1) = g(n, succ(n)).
As g is primitive recursive then so is f (Unit 2 Problem 1.4).

Question 12

[[This is based on the 2004 solution by the late Wilson Stother.]]

(i) (2 marks)

For any natural number n, define the URM program I(n) by
(1) C(1,n).
This implements the identity function *id* since it halts with the original value in register 1.
If Q and I(n) are concatenated then the resulting program will still compute the function pred. Thus there are infinitely many such programs which will compute pred.

(ii) (5 marks)

Recover the program P from the number e, and determine the maximum register number ρ(P) used by P. Now create P* by concatenating the following programs
(1) C(1, ρ(P)+1),
P, and
(1) C(ρ(P)+1,1)
(2) Z(2,ρ(Q)) [[Clear remaining registers used by Q]]
Q.

If f_P^1 is total, then P* saves the input in a register not used by P, executes P, and, as f_P^1 is total, moves on to the last instruction of P* which restores the original value of register 1. Thus P* computes the function pred in this case.

If f_P^1 is not total, then P* will not halt for some input n. For this input, P* will execute the first instruction. This does not affect register 1, so the program P will not halt. Thus, P* will *not* halt for this input. Hence the function computed by P* is not total, and so P* cannot compute the function pred.

(iii) (4 marks)

Suppose that X is recursive. Then there is an algorithm for testing whether a code number e* is in X.

Tot is the set of code numbers of URMs which compute total functions of one variable. We now have the following algorithm for deciding whether a given integer e is in *Tot*.

> There is a simple algorithm to check whether e is the code number of a URM.
> If not, then e is not in *Tot*.
> Otherwise, we can recover P from e, and construct P* as in (ii).
> Let P* have code number e*.
> From part (ii), $e \in Tot$ if and only if $e^* \in X$.

The existence of such an algorithm contradicts Theorem 3.2, so our assumption must be false. Hence X is not recursive.

Question 13

(i) [[See Unit 4 Example 3.3 and Problems 3.2 and 3.3. Anything starting with a universal qualifier (∀ or ∃) has to be a subformula]]

Let φ be the subformula $\forall x\ x = y'$; ψ be $\exists y\ (\forall x\ x = y'\ \&\ x = y')$; and χ be $x = y'$.
The given formula can then be written as $((\varphi \to \neg(\psi \vee \neg\chi)) \to (\varphi \to \chi))$
A truth table for this formula is

φ	ψ	χ	((φ → ¬(ψ ∨ ¬ χ)) → (φ → χ))
1	1	1	1 0 0 1 1 0 1 1 1 1 1
1	1	0	1 0 0 1 1 1 0 1 1 0 0
1	0	1	1 1 1 0 0 0 1 1 1 1 1
1	0	0	1 0 0 0 1 1 0 1 1 0 0
0	1	1	0 1 0 1 1 0 1 1 0 1 1
0	1	0	0 1 0 1 1 1 0 1 0 1 0
0	0	1	0 1 1 0 0 0 1 1 0 1 1
0	0	0	0 1 0 0 1 1 0 1 0 1 0
			(4) (3) (2)(1) (5) (1)

Since column 5 is all ones then the formula takes the truth value 1 under all interpretations and so the formula is a tautology.

(ii)(a)

Line	1	2	3	4	5	6	7	8	9
Ass.	1	2	2	4	2,4	2, 4	1, 2	2	1, 2

(ii)(b)

$(((\phi \vee \theta)\ \&\ (\neg\psi\ \&\ \phi)) \to (\neg\psi\ \&\ \phi))$.

(ii)(c)

(A) NO (B) YES

(iii)
As y cannot be freely substituted for x in $\exists y\ (y + \mathbf{0}') = x$ then the proof is not valid.

[[Linda Brown pointed out that my original solution was wrong. After seeing her solution I have come up with the following.]]

Let the interpretation $\mathcal{Z}_2$ have domain $Z_2 = \{0, 1\}$ where ', + and . are the functions given by the following tables.

x	0	1
x '	1	0

+	0	1
0	0	1
1	1	0

.	0	1
0	0	0
1	0	1

The assumption is true under this interpretation since for all x there is a y such that $y + \mathbf{0}' = x$.

[[If x = 0 then y = 1 and if x = 1 then y = 0]]

However there is no value of y such that $y + \mathbf{0}' = y$.

Therefore $\exists y\,(y + \mathbf{0}') = y$ is not a logical consequence of the assumption.

[[Linda's solution had the domain as the set of real numbers. I think having the domain as the set of rational numbers or integers would be other reasonable choices.]]

Question 14

(i) [[Only free occurrences of x are in the term $\exists t\, (y \,.\, x) = t$]]

(a) NO [[z becomes bound]]
(b) NO [[t becomes bound]]
(c) YES

(ii)(a)

1	(1)	$\exists y\, \forall x\, (x + y') = (y' \,.\, t)$	Ass
2	(2)	$\forall x\, (x + y') = (y' \,.\, t)$	Ass
2	(3)	$(x' + y') = (y' \,.\, t)$	UE, 2
2	(4)	$\exists y\, (x' + y) = (y \,.\, t)$	EI, 3
1	(5)	$\exists y\, (x' + y) = (y \,.\, t)$	EH, 4
1	(6)	$\exists x\, \exists y\, (x' + y) = (y \,.\, t)$	EI, 5

Therefore $\exists y\, \forall x\, (x + y') = (y' \,.\, t) \vdash \exists x\, \exists y\, (x' + y) = (y \,.\, t)$.

(ii)(b) [[I find Unit 6, Techniques for finding formal proofs useful]]

1	(1)	$(\phi \,\&\, \forall x(\neg\psi \vee \theta))$	Ass
1	(2)	$\forall x(\neg\psi \vee \theta)$	Taut, 1
1	(3)	$(\neg\psi \vee \theta)$	UE, 2
4	(4)	$\exists x\, (\neg\theta \leftrightarrow \phi)$	Ass
5	(5)	$(\neg\theta \leftrightarrow \phi)$	Ass
6	(6)	$\forall x(\psi \,\&\, \chi)$	Ass.

[[Here we have assumed the opposite of what we are trying to prove. Since the condition we are going to use is that ϕ does not contain a free occurrence of x then we are aiming to derive the contradiction $(\phi \,\&\, \neg\phi)$ and then use EH. From (1) we know ϕ is true so we have to show $\neg\phi$ is also true.]]

6	(7)	$(\psi \,\&\, \chi)$	UE, 6
1, 6	(8)	θ	Taut, 3, 7
1,5,6	(9)	$\neg\phi$	Taut, 5, 8
1,5,6	(10)	$(\phi \,\&\, \neg\phi)$	Taut, 1, 9
1,4,6	(11)	$(\phi \,\&\, \neg\phi)$	EH, 10
1,4	(12)	$(\forall x(\psi \,\&\, \chi) \rightarrow (\phi \,\&\, \neg\phi))$	CP, 11
1,4	(13)	$\neg\forall x(\psi \,\&\, \chi)$	Taut, 12
1	(14)	$(\exists x\, (\neg\theta \leftrightarrow \phi) \rightarrow \neg\forall x(\psi \,\&\, \chi))$	CP, 13

Therefore $(\phi \,\&\, \forall x(\neg\psi \vee \theta)) \vdash (\exists x\, (\neg\theta \leftrightarrow \phi) \rightarrow \neg\forall x(\psi \,\&\, \chi))$.

The assumption that x does not occur free in ϕ is required for the use of EH on line (11). [[Note that assumption 1 also contains ϕ.]]

Question 15

[[Expect two are theorems and one is not. Look for 2 similar sentences. Does one imply the other? (ii) implies (iii) so expect (ii) is not a theorem.]]

(i) [[Looks as if both sides of the equation equal (0.x).]]

-	(1)	$(\mathbf{0} . (x + \mathbf{0})) = (\mathbf{0} . (x + \mathbf{0}))$	II
2	(2)	$\forall x\ (x + \mathbf{0}) = x$	Ass. Q4
2	(3)	$(x + \mathbf{0}) = x$	UE, 2
2	(4)	$(\mathbf{0} . (x + \mathbf{0})) = (\mathbf{0} . x)$	Sub, 1, 3
-	(5)	$((\mathbf{0} + \mathbf{0}) . x)) = ((\mathbf{0} + \mathbf{0}) . x))$	II
2	(6)	$(\mathbf{0} + \mathbf{0}) = \mathbf{0}$	UE, 2
2	(7)	$(\mathbf{0} . x) = ((\mathbf{0} + \mathbf{0}) . x))$	Sub, 5, 6
2	(8)	$(\mathbf{0} . (x + \mathbf{0})) = ((\mathbf{0} + \mathbf{0}) . x))$	Sub, 4, 7
2	(9)	$\forall x\ (\mathbf{0} . (x + \mathbf{0})) = ((\mathbf{0} + \mathbf{0}) . x))$	UI, 8

As the assumption is axiom Q4 of Q then the sentence is a theorem of Q.

(ii) [[I always try non-standard interpretation $\mathcal{N}^{**}$ with x and y as α or β.]]

In the non-standard interpretation $\mathcal{N}^{**}$ let $x = \alpha$, and $y = \alpha$. Then

$((x + y) . x) = ((\alpha + \alpha) . \alpha) = (\beta . \alpha) = \alpha$, and
$(x . (x + y)) = (\alpha . (\alpha + \alpha)) = (\alpha . \beta) = \beta$.

All the axioms of Q hold in $\mathcal{N}^{**}$.
As $\forall x \forall y\ ((x + y) . x) = (x . (x + y))$ does not hold in the interpretation $\mathcal{N}^{**}$ then, it follows by the Correctness Theorem, the sentence is not a theorem of Q.

(iii) [[When have $\exists y$ then hopefully the sentence will be true when y = 0.]]

-	(1)	$((x + \mathbf{0}) . x) = ((x + \mathbf{0}) . x)$	II
2	(2)	$\forall x\ (x + \mathbf{0}) = x$	Ass. Q4
2	(3)	$(x + \mathbf{0}) = x$	UE, 2
2	(4)	$((x + \mathbf{0}) . x) = (x . x)$	Sub, 1, 3
-	(5)	$(x . (x + \mathbf{0})) = (x . (x + \mathbf{0}))$	II
2	(6)	$(x . x) = (x . (x + \mathbf{0}))$	Sub, 5, 3
2	(7)	$((x + \mathbf{0}) . x) = (x . (x + \mathbf{0}))$	Sub, 4, 6
2	(8)	$\forall x\ ((x + \mathbf{0}) . x) = (x . (x + \mathbf{0}))$	UI, 7
2	(9)	$\exists y\ \forall x\ ((x + y) . x) = (x . (x + y))$	EI, 8

As the assumption is axiom Q4 of Q then the sentence is a theorem of Q.

Question 16

(i) 2005 Solution by Linda Brown.
No such theory exists.
Suppose that theory T is not consistent but has an interpretation.
Hence there is a sentence of T, Φ say, such that $\vdash_T \Phi$ and $\vdash_T \neg \Phi$,
i.e. both Φ and ¬ Φ are theorems of T

By the Correctness Theorem both Φ and ¬ Φ are true in every interpretation in which the sentences of T are true and so must be true in the interpretation of T

However a sentence cannot be both true and false in the same interpretation and this contradicts our original supposition. Hence a theory which has an interpretation is consistent.

(ii) **No such theory exists.**

If T is a consistent theory which extends Q then, by Unit 8 Theorem 3.1, 0 = 1 is not a theorem of T.

(iii) **Such a theory exists.**

Let T be the extension of Q formed by adding the axiom
$\forall x \, \forall y \, (x + y) = (y + x)$
As the standard interpretation $\mathcal{N}$ is an interpretation of T then by part (i) T is consistent and
$\vdash_T \forall x \, \forall y \, (x + y) = (y + x)$

A better solution by Linda Brown is
CA is an example of a consistent theory which extends Q and has ∀x ∀y (x + y) = (y + x) as a theorem.

(iv) **Such a theory does not exist.**

Let A = {0, 3, 7}. A characteristic function for the set A is $\chi_A(n) = \begin{cases} 1, \textit{if } n=0 \\ 1, \textit{if } n=3 \\ 1, \textit{if } n=7 \\ 0, \textit{otherwise} \end{cases}$ so A is a recursive set. As T is a theory which extends Q then, by Unit 8 Theorem 1.1, A is representable in T.

(iv) **Such a theory does not exist.**

Gödel's First Incompleteness theorem states that there is no complete and consistent recursively axiomatizable extension of Q.