## *Question 9*

**(i) (a)**

| Next Instruction | $R_1$ | $R_2$ | $R_3$ |
|---|---|---|---|
| 1 | 2 | 1 | 1 |
| 2 | 2 | 1 | 1 |
| 3 | 2 | 2 | 1 |
| 4 | 2 | 3 | 1 |
| 5 | 2 | 3 | 2 |
| 1 | 2 | 3 | 2 |
| 6 | 2 | 3 | 2 |
| STOP | 3 | 3 | 2 |

**(i)(b)**

$f_P^1: \mathbb{N} \rightarrow \mathbb{N}$ is defined by $f_P^1(n) = 2n$.
$f_P^2: \mathbb{N}^2 \rightarrow \mathbb{N}$ is defined by $f_P^2(n, m) = m + 2n$.
$f_P^3: \mathbb{N}^3 \rightarrow \mathbb{N}$ is defined by $f_P^3(n, m, q) = \begin{cases} m + 2(n - q), & \text{if } n \geq q \\ \textit{undefined}, & \text{otherwise} \end{cases}$.

[[ Check that when we put q = 0 in $f_P^3$ we get $f_P^2$, and when we put m = 0 in $f_P^2$ we get $f_P^1$. ]]

**(ii)**

[[ Since $R_2$ is initially zero then it will be even at statement 1 and odd at statement 3. ]]

1 J(1, 2, 6) [[ Is $R_1 = R_2$ and $R_2$ even? ]]
2 S(2)
3 J(1, 2, 7) [[ Is $R_1 = R_2$ and $R_2$ odd? ]]
4 S(2)
5 J(1, 1, 1)
6 Z(2) [[ $R_1$ is even so set answer to zero ]]
7 C(2,1)

[[ In statement 3 we could jump to a statement after 7 as $R_1$ already contains n ]]

***Question 10***

**(i)** [[ See Unit 2, Problem 1.9 ]]

Let $\chi_{\leq}: \mathbb{N}^2 \to \mathbb{N}$ be defined by $\chi_{\leq}(n, m) = \overline{sg}\,(n \dot{-} m)$.

If $n > m$ then $n \dot{-} m > 0$ and so $\chi_{\leq}(n, m) = 0$.
If $n \leq m$ then $n \dot{-} m = 0$ and so $\chi_{\leq}(n, m) = 1$.

Since $\chi_{\leq}$ is obtained by substitution from the total primitive recursive functions $\overline{sg}$ and $\dot{-}$ then $\chi_{\leq}$ is also a total primitive recursive function.

Therefore $\chi_{\leq}$ is a characteristic function for the relation $\leq$ and so it is primitive recursive.

**(ii)** [[ See proof of Unit 2, Theorem 1.3. ]]

Since $R_1$ and $R_2$ are primitive recursive relations then their characteristic functions $\chi_{R_1}$ and $\chi_{R_2}$, respectively, are primitive recursive functions.

Consider the function

$$f(n_1, n_2) = g_1(n_1, n_2)\chi_{R_1}(n_1, n_2) + g_2(n_1, n_2)\chi_{R_2}(n_1, n_2).$$

If $\chi_{R_1}(n_1, n_2) = 1$ then, since the relations $R_1$ and $R_2$ are mutually exclusive $\chi_{R_2}(n_1, n_2) = 0$. So, in this case, $f(n_1, n_2) = g_1(n_1, n_2)$.

If $\chi_{R_1}(n_1, n_2) = 0$ then, since the relations $R_1$ and $R_2$ are mutually exclusive and exhaustive, $\chi_{R_2}(n_1, n_2) = 1$. So, in this case, $f(n_1, n_2) = g_2(n_1, n_2)$.

Since the function f is obtained from the primitive recursive functions mult, add, $\chi_{R_1}$ and $\chi_{R_2}$ by substitution then f is primitive recursive.

**(iii)** Use of Unit 2 Theorem 1.5

Define the functions

$g_1(n, m) = n^8 = \exp(n, 8)$
$g_2(n, m) = mn = \text{mult}(m, n)$,
$g_3(n, m) = 9 = C_9^2\ (n, m)$,

and the relations

$R_1(n, m) \Leftrightarrow \chi_O(n + m + 4)$,
$R_2(n, m) \Leftrightarrow 3n + 5m = 8000$,
$R_3(n, m) \Leftrightarrow$ not $R_1(n, m)$ and not $R_2(n, m)$.

Then we can write

$$f(n, m) = \begin{cases} g_1(n,m) & \text{if } R_1(n,m) \\ g_2(n,m) & \text{if } R_2(n,m) \\ g_3(n,m) & \text{if } R_3(n,m) \end{cases}$$

As $g_1$, $g_2$, and $g_3$ can be written the primitive recursive functions $C_9^2$, mult, and exp using constants then $g_1$, $g_2$, and $g_3$ are primitive recursive functions.

The characteristic function of the relation $R_1$, $\chi_{R1}(n, m) = \chi_O(m + n + 4)$. As $\chi_{R_1}$ is obtained by substitution from the primitive recursive functions $\chi_O$, and add using constants, then it is a primitive recursive function. Hence $R_1$ is a primitive recursive relation.

The characteristic function of the relation $R_2$, $\chi_{R2}(n,m) = \chi_{eq}(3n + 5m, 8000)$. As $\chi_{R_2}$ is obtained by substitution from the primitive recursive functions $\chi_{eq}$, mult and add using constants, then it is a primitive recursive function. Hence $R_2$ is a primitive recursive relation.

Using the result of Unit 2, Problem 1.10, then $R_3$ is also a primitive recursive relation.

From the definition of $R_3$ it follows that the set of relations $R_1$, $R_2$, and $R_3$ are exhaustive.

If the relation $R_1$ holds then one of n and m is odd and the other even. If the relation $R_2$ holds then n and m are both even or both odd. Therefore $R_1$ and $R_2$ are mutually exclusive. From the definition of $R_3$, if the relation $R_3$ holds then neither $R_1$ or $R_2$ holds. Therefore $R_1$, $R_2$ and $R_3$ are mutually exclusive.

Since all the conditions required for the use of Theorem 1.5 of Unit 2 hold then it follows that f is primitive recursive.

## Question 11

**(i)(a)**

g(n1, n2) = f ( $U_1^2(n_1, n_2)$ , succ(zero($U_1^2(n_1, n_2)$)), $U_2^2(n_1, n_2)$ , ).

As g is obtained by substitution from the primitive recursive function f and the basic primitive recursive functions succ, zero , $U_1^2$ , and $U_2^2$ , then g is a primitive recursive function.

[[ This is similar to Unit 2, Problem 1.3. However we would also have to prove $C_1^2$ is primitive recursive to use that answer. ]]

**(i)(b)**

Let exp(n, 0) = f(n), where f(n) = succ(zero(n)).
and exp(n, m + 1) = g(n, m, exp(n, m))
where g(n1, n2, n3) = mult( $U_1^3(n_1, n_2, n_3)$, $U_3^3(n_1, n_2, n_3)$ ).

As f is obtained by substitution from the primitive recursive functions succ and zero then it is primitive recursive.

g is a primitive recursive function since it is obtained by substitution from mult and the basic primitive recursive functions $U_1^3$ , and $U_3^3$ . As exp is formed by primitive recursion from the primitive recursive functions f and g, then exp is a primitive recursive function.

**(ii)(b) 4 marks**

*Based on the 2004 solution by Lisette Petrie.*

$f(n) = \mu y(n < y$ and the remainder of $y$ on division by 5 is 1)
Consider the relation $T$ given by
$T(n, y) \Leftrightarrow n < y$ and the remainder of $y$ on division by 5 is 1.
The relation < is primitive recursive [*HB p21*]. The characteristic function of the relation "the remainder of *y* on division by 5 is 1" is $\chi_R(y)$ = eq(rem(y, 5), 1). Since it is obtained by substitution from the primitive recursive functions eq and rem using constants then it is primitive recursive.
Then $\chi_T(n, y) = \chi_<(n, y)$ and $\chi_R(y)$,
which is obtained by substitution from the primitive recursive functions $\chi_<$ and $\chi_R$ and so is primitive recursive [*HB p21 result of problem 1.10*].
By Theorem 3.5 *[HB p23]* the function $g : \mathbb{N}^2 \to \mathbb{N}$ given by
$$g(n, z) = \mu y \leq z\ T(n, y)$$
is primitive recursive.
A suitable bound on $y$ in terms of $n$ is $n + 5$, so
$$f(n) = \mu y \leq (n + 5)\ T(n, y) = g(n, n + 5)$$
which is a substitution of primitive recursive functions using constants, so is primitive recursive.

## Question 12

A copy of the 2004 solution by the late Wilson Stother.

**(i)** (2 marks)

For any natural number $n$, define the URM program $I(n)$ by
(1) $C(1,n)$.
This implements the identity function *id* since it halts with the original value in register 1.
Thus there are infinitely many such programs.

**(ii)** (5 marks)

Given a program P, first determine the maximum register number $\rho(P)$ used by P.
Now create P* by concatenating the following programs
(1) $C(1, \rho(P)+1)$,
P, and
(1) $C(\rho(P)+1,1)$.

If $f_P^1$ is total, then P* saves the input in a register not used by P, executes P, and, as $f_P^1$ is total, moves on to the last instruction of P* which restores the original value of register 1.
Thus P* computes the identity function *id* in this case.

If $f_P^1$ is not total, then P* will not halt for some input $n$. For this input, P* will execute the first instruction. This does not affect register 1, so the program P will not halt. Thus, P* will *not* halt for this input. Hence the function computed by P* is not total, and so P* cannot compute the identity function.

**(iii)** (4 marks)

Suppose that X is recursive. Then there is an algorithm for testing whether a code number $e^*$ is in X.

*Tot* is the set of code numbers of URMs which compute total functions of one variable.
We now have the following algorithm for deciding whether a given integer $e$ is in *Tot*.

> There is a simple algorithm to check whether $e$ is the code number of a URM.
> If not, then $e$ is not in *Tot*.
> Otherwise, we can recover P from $e$, and construct P* as in (ii).
> Let P* have code number $e^*$.
> From part (ii), $e \in Tot$ if and only if $e^* \in X$.

The existence of such an algorithm contradicts Theorem 3.2, so our assumption must be false.

Hence X is not recursive.

## Question 13

(i) [[ See Unit 4 Example 3.3 and Problems 3.2 and 3.3 ]]

Let $\varphi$ be the subformula $\exists x\ x' = \mathbf{0}$; $\psi$ be $x' = \mathbf{0}$; and $\chi$ be $\forall x(x' = \mathbf{0} \leftrightarrow \exists x\ x' = \mathbf{0})$.
The given formula can then be written as $((\varphi \to \neg\psi) \to ((\chi \,\&\, \psi) \to \neg\varphi))$
A truth table for this formula is

| $\varphi$ | $\psi$ | $\chi$ | $((\varphi \to \neg\ \psi) \to ((\chi \,\&\, \psi) \ \to \neg\ \varphi))$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 0 0 1 1 1 1 1 0 0 1 |
| 1 | 1 | 0 | 1 0 0 1 1 0 0 1 1 0 1 |
| 1 | 0 | 1 | 1 1 1 0 1 1 0 0 1 0 1 |
| 1 | 0 | 0 | 1 1 1 0 1 0 0 0 1 0 1 |
| 0 | 1 | 1 | 0 1 0 1 1 1 1 1 1 1 0 |
| 0 | 1 | 0 | 0 1 0 1 1 0 0 1 1 1 0 |
| 0 | 0 | 1 | 0 1 1 0 1 1 0 0 1 1 0 |
| 0 | 0 | 0 | 0 1 1 0 1 0 0 0 1 1 0 |
| | | | (2) (1) (3) (1) (2) (1) |

Since column 3 is all ones then the formula takes the truth value 1 under all interpretations then the formula is a tautology.

**(ii)(a)**

| **Line** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **Ass.** | 1 | 2 | 1 | 4 | 2 | 1, 2 | 2 | 1, 2 | 1, 4 |

**(ii)(b)**

$(((\theta \to \neg\,\psi) \,\&\, (\neg\,\phi \,\&\, \psi)) \to (\neg\,\phi \,\&\, \neg\,\psi))$.

**(ii)(c)**

(A) YES (B) NO

**(iii)**

As the assumption on line 1 contains a free occurrence of the variable x then the use of the UI rule is invalid.

Let U be the interpretation with domain $\{0, \delta\}$. Define the operation · by

| **·** | **0** | **δ** |
|---|---|---|
| **0** | 0 | δ |
| **δ** | δ | δ |

In this interpretation statement $(x\ .\ \mathbf{0}) = \mathbf{0}$ is true when $x = \mathbf{0}$. However $\forall(x\ .\ \mathbf{0}) = \mathbf{0}$ is false.
Therefore $\forall(x\ .\ \mathbf{0}) = \mathbf{0}$ is not a logical consequence of $(x\ .\ \mathbf{0}) = \mathbf{0}$.

### *Question 14*

**(i)** [[ Only free occurrences of x are in the term $\exists y\,(y'.z) = (x.t)$ ]]

(a) NO [[ y becomes bound ]]
(b) NO [[ z becomes bound ]]
(c) YES

**(ii)(a)**

| | | | |
|---|---|---|---|
| 1 | (1) | $\exists y\,\forall x\,(x\,.\,y') = x$ | Ass |
| 2 | (2) | $\forall x\,(x\,.\,y') = x$ | Ass |
| 2 | (3) | $(x'\,.\,y') = x'$ | UE, 2 |
| 2 | (4) | $\exists y\,(x'\,.y) = x'$ | EI, 3 |
| 2 | (5) | $\forall x\,\exists y\,(x'\,.y) = x'$ | UI, 4 |
| 1 | (6) | $\forall x\,\exists y\,(x'\,.y) = x'$ | EH, 5 |

Therefore $\exists y\,\forall x\,(x\,.\,y') = x \vdash \forall x\,\exists y\,(x'\,.y) = x'$.

**(ii)(b)** [[ I find Unit 6, Techniques for finding formal proofs useful ]]

| | | | |
|---|---|---|---|
| 1 | (1) | $(\phi\ \&\ \forall x(\neg\phi \vee \psi))$ | Ass |
| 2 | (2) | $\exists x\,\neg\theta$ | Ass |
| 3 | (3) | $\neg\theta$ | Ass |
| 4 | (4) | $\forall x(\psi\rightarrow\theta)$ | Ass. [[Contradiction]] |
| 4 | (5) | $(\psi\rightarrow\theta)$ | UE, 4 |
| 3,4 | (6) | $\neg\psi$ | Taut, 3, 5 |
| 1 | (7) | $\forall x(\neg\phi \vee \psi)$ | Taut, 1 |
| 1 | (8) | $(\neg\phi \vee \psi)$ | UE, 7 |
| 1,3,4 | (9) | $\neg\phi$ | Taut, 6, 8 |
| 1 | (10) | $\phi$ | Taut, 1 |
| 1,3,4 | (11) | $(\phi\ \&\ \neg\phi)$ | Taut, 9, 10 |
| 1,2,4 | (12) | $(\phi\ \&\ \neg\phi)$ | EH, 11 |
| 1,2 | (13) | $(\forall x(\psi\rightarrow\theta) \rightarrow (\phi\ \&\ \neg\phi))$ | CP, 12 |
| 1,2 | (14) | $\neg\forall x(\psi\rightarrow\theta)$ | Taut, 13 |
| 1 | (15) | $(\exists x\,\neg\theta \rightarrow \neg\forall x(\psi\rightarrow\theta))$ | CP, 14 |

Therefore $(\phi\ \&\ \forall x(\neg\phi \vee \psi)) \vdash (\exists x\,\neg\theta \rightarrow \neg\forall x(\psi\rightarrow\theta))$.

The assumption that x does not occur free in $\phi$ is required for the use of EH on line (12). [[Note that assumption 1 also contains $\phi$. ]]

### *Question 15*

**(i)** [[ Looks as if both sides of the equation equal (0.x). ]]

| | | | |
|---|---|---|---|
| - | (1) | $((\mathbf{0}\ .\ \mathbf{0})\ .\ x) = ((\mathbf{0}\ .\ \mathbf{0})\ .\ x))$ | II |
| 2 | (2) | $\forall x\ (x\ .\mathbf{0}) = \mathbf{0}$ | Ass. Q6 |
| 2 | (3) | $(\mathbf{0}\ .\mathbf{0}) = \mathbf{0}$ | UE, 2 |
| 2 | (4) | $((\mathbf{0}\ .\ \mathbf{0})\ .\ x) = (\mathbf{0}\ .\ x)$ | Sub, 1, 3 |
| - | (5) | $((\mathbf{0}\ .\ x) + (x\ .\ \mathbf{0})) = ((\mathbf{0}\ .\ x) + (x\ .\ \mathbf{0}))$ | II |
| 2 | (6) | $(x\ .\mathbf{0}) = \mathbf{0}$ | UE, 2 |
| 2 | (7) | $((\mathbf{0}\ .\ x) + \mathbf{0}) = ((\mathbf{0}\ .\ x) + (x\ .\ \mathbf{0}))$ | Sub, 5, 6 |
| 8 | (8) | $\forall x\ (x + \mathbf{0}) = x$ | Ass. Q4 |
| 8 | (9) | $((\mathbf{0}\ .\ x) + \mathbf{0}) = (\mathbf{0}\ .\ x)$ | UE, 2 |
| 2, 8 | (10) | $(\mathbf{0}\ .\ x) = ((\mathbf{0}\ .\ x) + (x\ .\ \mathbf{0}))$ | Sub, 7, 9 |
| 2, 8 | (11) | $((\mathbf{0}\ .\ \mathbf{0})\ .\ x) = ((\mathbf{0}\ .\ x) + (x\ .\ \mathbf{0}))$ | Sub, 4, 10 |
| 2, 8 | (12) | $\forall x\ ((\mathbf{0}\ .\ \mathbf{0})\ .\ x) = ((\mathbf{0}\ .\ x) + (x\ .\ \mathbf{0}))$ | UI, 11 |

As the assumptions are axioms Q4 and Q6 of Q then the sentence is a theorem of Q.

**(ii)** [[ If this is a theorem then so is (iii). Therefore unlikely to be one. ]]

In $\mathcal{N}^{**}$ let x = α, and y = α. Then

$((x.y).x) = ((\alpha.\alpha).\alpha) = (\beta.\alpha) = \alpha$, and

$(x.(y.x)) = (\alpha.(\alpha.\alpha)) = (\alpha.\beta) = \beta$.

All the axioms of Q hold in $\mathcal{N}^{**}$. As $\forall x \forall y((x.y).x = x.(y.x))$ does not hold in the interpretation $\mathcal{N}^{**}$ then, it follows by the Correctness Theorem, the sentence is not a theorem of Q.

(iii)

| | | | |
|---|---|---|---|
| - | (1) | $((\mathbf{0}.y).\ \mathbf{0}) = ((\mathbf{0}.y).\ \mathbf{0})$ | II |
| 2 | (2) | $\forall x\ (x.\ \mathbf{0}) = \mathbf{0}$ | Ass. Q6 |
| 2 | (3) | $((\mathbf{0}.y).\ \mathbf{0}) = \mathbf{0}$ | UE, 2 |
| 2 | (4) | $((\mathbf{0}.y).\ \mathbf{0}) = \mathbf{0}$ | Sub, 1, 3 |
| - | (5) | $(\mathbf{0}.(y.\ \mathbf{0})) = (\mathbf{0}.(y.\ \mathbf{0}))$ | II |
| 2 | (6) | $(y.\ \mathbf{0}) = \mathbf{0}$ | UE, 2 |
| 2 | (7) | $(\mathbf{0}.\ \mathbf{0}) = (\mathbf{0}.(y.\ \mathbf{0}))$ | Sub, 5, 6 |
| 2 | (8) | $(\mathbf{0}.\ \mathbf{0}) = \mathbf{0}$ | UE, 2 |
| 2 | (9) | $\mathbf{0} = (\mathbf{0}.(y.\ \mathbf{0}))$ | Sub, 7, 8 |
| 2 | (10) | $((\mathbf{0}.y).\ \mathbf{0}) = (\mathbf{0}.(y.\ \mathbf{0}))$ | Sub, 4, 9 |
| 2 | (11) | $\forall y\ ((\mathbf{0}.y).\ \mathbf{0}) = (\mathbf{0}.(y.\ \mathbf{0}))$ | UI, 10 |
| 2 | (12) | $\exists x\ \forall y\ ((x.y).\ x) = (x.(y.\ x))$ | EI, 11 |

As the assumption is axiom Q6 of Q then the sentence is a theorem of Q.

## Question 16

**(i)** 2005 Solution by Linda Brown.

Suppose that theory T is not consistent but has an interpretation.
Hence there is a sentence of T, $\Phi$ say, such that $\vdash_T \Phi$ and $\vdash_T \neg\Phi$,
i.e. both $\Phi$ and $\neg\Phi$ are theorems of T

By the Correctness Theorem both $\Phi$ and $\neg\Phi$ are true in every interpretation in which the sentences of T are true and so must be true in the interpretation of T

However a sentence cannot be both true and false in the same interpretation and this contradicts our original supposition.

Hence a theory which has an interpretation is consistent.

**(ii)(a) True**

Let A be the set of the Gödel numbers of the infinite number of axioms of PA.

Given any number n then we can write an algorithm to determine whether n is the Gödel number of a formula. If it is not then return 0. If it is and equals the Gödel number of any of the axioms of Q then return 1. By processing the Gödel number we can determine whether it has the form of an induction axiom. If it has return 1 otherwise return 0.

Therefore, by Church's Thesis, the characteristic function of the set A is recursive, and so the set A is recursive. Therefore PA is recursively axiomatizable.

**(ii)(b) True**

Assume that PA is not consistent. Therefore, by definition, there is a sentence $\Phi$ of the formal language such that both $\vdash_{PA} \Phi$ and $\vdash_{PA} \neg\Phi$.

The Correctness Theorem tells us that if a formula is derivable from PA that it is a logical consequence of PA providing that there is an interpretation of PA.

Since $\mathcal{N}$ is an interpretation of PA then both $\Phi$ and $\neg\Phi$ are logical consequences of PA.
Since $\Phi$ cannot be both true and false in an interpretation then our assumption that PA is not consistent must be incorrect. So PA is consistent and the statement is true.

**(ii)(c) False**

By Unit 7, Theorem 2.1(b) $\vdash_Q \mathbf{0} \neq \mathbf{1}$. So $\vdash_Q \neg\, \mathbf{0} = \mathbf{1}$. Since PA is an extension of Q then theorems of Q are also theorems of PA. So $\vdash_{PA} \neg\, \mathbf{0} = \mathbf{1}$.

Since PA is consistent, by part (ii), then both $\vdash_{PA} \mathbf{0} = \mathbf{1}$ and $\vdash_{PA} \neg\, \mathbf{0} = \mathbf{1}$ cannot be true.

Therefore $\vdash_{PA} \mathbf{0} = \mathbf{1}$ is not true.

**OR**

| | | | |
|---|---|---|---|
| 1 | (1) | $\forall x \neg\, \mathbf{0} = x'$ | Ass |
| 1 | (2) | $\neg\, \mathbf{0} = \mathbf{0}'$ | UE, 1 |

As the assumption is an axiom of PA then $\vdash_{PA} \neg\, \mathbf{0} = \mathbf{1}$.

Since PA is consistent ….

**(ii)(d) False**

Gödel's First Completeness theorem states that there is no complete and consistent recursively axiomatizable extension of Q.

Since PA is an extension of Q and, as we have shown in parts (i) and (ii), it is consistent and recursively axiomatizable this means PA cannot be complete.