Question 9

(i) (a)

Next Instruction	R_1	R_2	R_3
1	2	0	0
2	2	1	0
3	2	1	0
4	2	1	1
1	2	1	1
2	2	2	1
5	2	2	1
STOP	1	2	1

(i)(b)

P does not compute the function f when the input is zero. In this case the contents of register 1 (0) is never equal to the contents of register 2 (≥ 1) when instruction 2 is executed. Therefore the program never terminates as the loop consisting of instructions 1-4 is executed ad-infinitum..

$f_P^1: \mathbb{N} \to \mathbb{N}$ is defined by $f_P^1(n) = \begin{cases} n-1, & \text{if } n>0 \\ \text{undefined}, & \text{otherwise} \end{cases}$.

$f_P^2: \mathbb{N}^2 \to \mathbb{N}$ is defined by $f_P^2(n, m) = \begin{cases} n-m-1, & \text{if } n>m \\ \text{undefined}, & \text{otherwise} \end{cases}$.

$f_P^3: \mathbb{N}^3 \to \mathbb{N}$ is defined by $f_P^3(n, m, p) = \begin{cases} n-m+p-1, & \text{if } n>m \\ \text{undefined}, & \text{otherwise} \end{cases}$.

[[Check that when we put $p = 0$ in f_P^3 we get f_P^2. Similarly put $m = 0$ in f_P^2.]]

(ii)

[[We need to put in a test for n = 0 to ensure that the program works in this case. As an instruction is added at the start the following statement numbers change so we also need to change the instruction numbers in the jump instructions.]]

1 J(1, 2, 6) [[Check for 0 and adjust statement nos. in Jumps]]
2 S(2)
3 J(1, 2, 6)
4 S(3)
5 J(1, 1, 2)
6 C(3, 1).

Question 10

(i) [[See Unit 2, Problem 1.9 for $\leq$.]]

Let $\chi_{\geq}: \mathbb{N}^2 \to \mathbb{N}$ be defined by $\chi_{\geq}(n, m) = \overline{sg}(m \dot{-} n)$.

If $n < m$ then $m \dot{-} n > 0$ and so $\chi_{\geq}(n, m) = 0$.
If $n \geq m$ then $m \dot{-} n = 0$ and so $\chi_{\geq}(n, m) = 1$.

Since $\chi_{\geq}$ is obtained by substitution from the total primitive recursive functions $\overline{sg}$ and $\dot{-}$ then $\chi_{\geq}$ is also a total primitive recursive function.

Therefore $\chi_{\geq}$ is a characteristic function for the relation $\geq$ and so it is primitive recursive.

(ii) [[Similar to proof of Unit 2, Theorem 1.2.]]

Since A is a primitive recursive set then its characteristic function χ_A is primitive recursive.

Let $\chi_{N^2 \setminus A}(n_1, n_2) = \overline{sg}(\chi_A(n_1, n_2))$.

Therefore the relation $\chi_{N^2 \setminus A}$ is primitive recursive since its characteristic function is obtained by substitution from the primitive functions $\overline{sg}$ and χ_A.

Therefore $\mathbb{N}^2 \setminus A$ is a primitive recursive set.

(iii) [[Use of Unit 2 Theorem 1.5. Similar to Unit 2, Additional Exercise, Sect. 1, Qu. 4]]

Define the functions

$g_1(n, m) = m^4 = \exp(m, 4)$
$g_2(n, m) = m + n = \text{add}(m, n)$,
$g_3(n, m) = 17 = C^2_{17}(n, m)$,

and the relations

$R_1(n, m) \Leftrightarrow \max(3n, 2m) = 300$,
$R_2(n, m) \Leftrightarrow 4n + 3m \geq 900$,
$R_3(n, m) \Leftrightarrow$ not $R_1(n, m)$ and not $R_2(n, m)$.

Then we can write

$$f(n, m) = \begin{cases} g_1(n, m) & \text{if } R_1(n, m) \\ g_2(n, m) & \text{if } R_2(n, m) \\ g_3(n, m) & \text{if } R_3(n, m) \end{cases}$$

As g_1, g_2, and g_3 can be written using the primitive recursive functions C^2_{17} , add, and exp using constants then g_1, g_2, and g_3 are primitive recursive functions.

The characteristic function of the relation R_1, $\chi_{R1}(n, m) = \chi_{eq}$ (max (3m, 2n), 300). As χ_{R_1} is obtained by substitution from the primitive recursive functions χ_{eq}, max, and mult using constants, then it is a primitive recursive function. Hence R_1 is a primitive recursive relation.

The characteristic function of the relation R_2, $\chi_{R2}(n, m) = \chi_{\geq}(4n + 3m, 900)$. As χ_{R_2} is obtained by substitution from the primitive recursive functions $\chi_{\geq}$, mult and add using constants, then it is a primitive recursive function. Hence R_2 is a primitive recursive relation.

Using the result of Unit 2, Problem 1.10, then R_3 is also a primitive recursive relation.

From the definition of R_3 it follows that the set of relations R_1, R_2, and R_3 are exhaustive.

If the relation R_1 holds then $n \leq 100$ and $m \leq 150$ and $4n + 3m \leq 400 + 450 < 900$. Therefore R_1 and R_2 are mutually exclusive. From the definition of R_3, if the relation R_3 holds then neither R_1 or R_2 holds. Therefore R_1, R_2 and R_3 are mutually exclusive.

Since all the conditions required for the use of Theorem 1.5 of Unit 2 hold then it follows that f is primitive recursive.

Question 11

(i)(a) [[Similar to Unit 2, Problem 1.3]]

$g(n_1, n_2) = f\,(U_2^2(n_1, n_2),\ succ\,(U_1^2(n_1, n_2)))$.

As g is obtained by substitution from the primitive recursive functions f, U_1^2, succ, and U_2^2, then g is a primitive recursive function.

(i)(b) [[Unit 2, Example 1.5.]]

Let $n \dot{-} 0 = n$, and $n \dot{-} (m + 1) = pred(n \dot{-} m)$.
Since $\dot{-}$ is defined by primitive recursion from the primitive recursive function pred then it is primitive recursive.

OR

Let cosf : $(n, m) \to n \dot{-} m$.
$cosf(n, 0) = f(n)$, where $f(n) = U_1^1(n)$.
$cosf(n, m + 1) = g(n, m, cosf(n, m))$, where $g(n, m, p) = pred(U_3^3(n, m, p))$.
Since U_1^1 is a primitive recursive function, and g is obtained by substitition from the primitive recursive functions pred and U_3^3 then f and g are primitive recursive functions.
Therefore cosf and $\dot{-}$ are primitive recursive functions.

(ii) [[Similar to Unit 2, Problem 3.4.]]

Let R be the relation given by $R(n, y) \Leftrightarrow n^4 < y^3$.

Its characterisic function $\chi_R(n, y) = \chi_<(exp(n, 4), exp(y, 3))$, is obtained by substitution from the primitive recursive functions $\chi_<$ and exp using constants. Hence it is primitive recursive.

By Theorem 3.5 the function $g : \mathbb{N}^2 \to \mathbb{N}$ given by
$g(n, z) = \mu y \le z\ R(n, y)$
is primitive recursive.

As $n^4 < y^3$ then a suitable bound on y in terms of n is $n^2 + 1$, since $n^4 < n^6 + 1 < (n^2 + 1)^3$.
[[I have added the 1 to cater for n = 0 and 1.]].
Let $h : \mathbb{N} \to \mathbb{N}$ given by $h(n) = succ\,(exp(n, 2))$. Therefore h is primitive recursive by substitution from the primitive recursive functions succ, and exp using constants.

$$f(n) = \mu y\ R(n, y) = g(n, h(n)).$$

As f is obtained from the primitive recursive functions g and h by substitution then f is primitive recursive.

Question 12

(i) [[Similar to Unit 3, Problem 3.3.]]

For any natural number n, define a URM program by

(1) C(1, 1)
⋮
(n) C(n, n)
(n+1) Z(1)
(n+2) S(1)
(n+3) S(1)

This implements the constant function C_2 since it halts with the value 2 in register 1. As a program exists for each $n \in \mathbb{N}$, where $n > 0$, then there are infinitely many such programs.

(ii) [[See Unit 3, Problem 3.4.]]

The program P^* can be created by concatenating the programs
P, and
(1) Z(1).
(2) S(1).
(3) S(1).

If f_P^1 is total, then P* executes P. As f_P^1 is total, the last three instructions of P* are also executed. These set the value of register 1 to 2. Therefore P* computes the constant function C_2. If f_P^1 is not total, then P will not halt for some input n, and neither will P* for the same input. So P* computes the constant function C_2 function precisely when the function f_P^1 is total.

(iii) [[See Unit 3, Problem 3.4.]]

Assume the set X of code numbers of URM programs which compute C_2 is recursive. We shall show that is X is recursive then so is *Tot*, the set of numbers that code URM programs which compute a total function of one variable.

Let e be any code number. Initially check to see whether e codes a URM program. This is possible since the set of code numbers, *Prog*, is primitive recursive [[HB p21]]. If e does not code a URM program then $e \notin$ *Tot*.

If e does code a URM program then the instructions of the program P can be recovered from e and the program P^* which computes the function C_2 can then be created as described in part (ii). The code number e^* of P^* can then be determined. As the set X is recursive then there is an algorithm for deciding if a number $e^* \in X$. As $e^* \in X$ if and only if $e \in$ *Tot* then we can determine whether $e \in$ *Tot*.

Theorem 3.2 of Unit 3 states that there is no algorithm which determines whether $e \in$ Tot. As we have found one then the assumption that the set X is recursive must be false.

Question 13

(i) [[Same as 2005 Qu 13 (i)]]

Let φ be the subformula $\exists y\, y' = x$; ψ be $\neg y' = x$; and χ be $\exists y\, (y' = x \leftrightarrow \exists y\, y' = x)$.

The given formula can then be written as $((\neg\varphi \to (\psi \,\&\, \chi)) \to (\psi \vee \varphi))$

A truth table for this formula is

φ	ψ	χ	$((\neg\varphi \to (\psi \,\&\, \chi)) \to (\psi \vee \varphi))$
1	1	1	0 1 1 1 1 1 1 1 1 1
1	1	0	0 1 1 1 0 0 1 1 1 1
1	0	1	0 1 1 0 0 1 1 0 1 1
1	0	0	0 1 1 0 0 0 1 0 1 1
0	1	1	1 0 1 1 1 1 1 1 1 0
0	1	0	1 0 0 1 0 0 1 1 1 0
0	0	1	1 0 0 0 0 1 1 0 0 0
0	0	0	1 0 0 0 0 0 1 0 0 0
			(1) (2) (1) (3) (1)

Since column 3 is all ones then the formula takes the truth value 1 under all interpretations.

(ii) [[Same as 2005 Qu 13 (ii)]]
(ii)(a)

Line	1	2	3	4	5	6	7	8	9
Ass.	1	1	3	4	1,3	1	1,3	1,4	1

(ii)(b)

$(((\phi \,\&\, \neg\psi) \,\&\, (\psi \to \theta)) \to (\psi \to \theta))$.

(ii)(c)

(A) YES (B) NO. [[ψ or θ may contain free occurrences of x]]

(iii) [[See Unit 5, Section 3.2, Problem 3.4.]]

As the variable v may have free occurrences in the formula on line 2 then the EH rule cannot be used on line 3.

Take the standard interpretation $\mathcal{N}$ with domain $\mathbb{N}$. In this interpretation $\exists v\, v = 0$ is true as there is a zero element in $\mathbb{N}$. Since there are non-zero elements in $\mathbb{N}$ then, for some v, we have $v = 0$ is false. Therefore $v = 0$ is not a logical consequence of $\exists v\, v = 0$.

Question 14

(i) [[Only free occurrence of x is in the term $\exists y\,(y+x) = y$]]

(a) YES
(b) NO [[y becomes bound]]
(c) NO [[t becomes bound]]

(ii)(a) [[Same as 2003, Qu. 14 (ii)(a)]]

[[This is a special case of Unit 5, Section 3.2, Example 3.6.]]

1	(1)	$\exists y\,\exists x\,(x + x) = y$	Ass
2	(2)	$\exists x\,(x + x) = y$	Ass
3	(3)	$(x + x) = y$	Ass
3	(4)	$\exists y(x + x) = y$	EI, 3
3	(5)	$\exists x\,\exists y\,(x + x) = y$	EI, 4
2	(6)	$\exists x\,\exists y\,(x + x) = y$	EH, 5
1	(7)	$\exists x\,\exists y\,(x + x) = y$	EH, 6

Therefore $\exists y\,\exists x\,(x + x) = y \vdash \exists x\,\exists y\,(x + x) = y$.

(ii)(b)

1	(1)	$\forall x\,(\theta \rightarrow (\neg\phi \vee \neg\psi))$	Ass
1	(2)	$(\theta \rightarrow (\neg\phi \vee \neg\psi))$	UE, 1
3	(3)	ψ	Ass
4	(4)	$\exists x\,(\phi\ \&\ \theta)$	Ass
5	(5)	$(\phi\ \&\ \theta)$	Ass
1, 5	(6)	$(\neg\phi \vee \neg\psi)$	Taut, 2, 5
1, 5	(7)	$\neg\psi$	Taut, 5, 6
1, 3, 5	(8)	$(\psi\ \&\ \neg\psi)$	Taut, 3, 7
1, 3, 4	(9)	$(\psi\ \&\ \neg\psi)$	EH, 8
1, 3	(10)	$(\exists x\,(\phi\ \&\ \theta) \rightarrow (\psi\ \&\ \neg\psi))$	CP, 9
1, 3	(11)	$\neg\exists x\,(\phi\ \&\ \theta)$	Taut, 10
1	(12)	$(\psi \rightarrow (\neg\exists x\,(\phi\ \&\ \theta)))$	CP, 11

The assumption that x does not occur free in ψ is required for the use of EH on line (9).

Question 15

[[Same as 2003, Qu 15 with parts (ii) and (iii) interchanged.]]

(i) [[Looks as if both sides of the equation equal ($\mathbf{0}$.x).]]

-	(1)	$(\mathbf{0}.(x + \mathbf{0})) = (\mathbf{0}.(x + \mathbf{0}))$	II
2	(2)	$\forall x\ (x + \mathbf{0}) = x$	Ass. Q4
2	(3)	$(x + \mathbf{0}) = x$	UE, 2
2	(4)	$(\mathbf{0}.(x + \mathbf{0})) = (\mathbf{0}.x)$	Sub, 1, 3
-	(5)	$((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0})) = ((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0}))$	II
6	(6)	$\forall x\ (x.\mathbf{0}) = \mathbf{0}$	Ass. Q6
6	(7)	$(\mathbf{0}.\mathbf{0}) = \mathbf{0}$	UE, 6
6	(8)	$((\mathbf{0}.x) + \mathbf{0}) = ((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0}))$	Sub, 5, 7
2	(9)	$((\mathbf{0}.x) + \mathbf{0}) = (\mathbf{0}.x)$	UE, 2
2,6	(10)	$(\mathbf{0}.x) = ((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0}))$	Sub, 8, 9
2,6	(11)	$(\mathbf{0}.(x + \mathbf{0})) = ((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0}))$	Sub, 4, 10
2,6	(12)	$\forall x\ (\mathbf{0}.(x + \mathbf{0})) = ((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0}))$	UI, 11

As the assumptions are axioms Q4 and Q6 of Q then the sentence is a theorem of Q.

(ii) [[(iii) $\Rightarrow$ (ii). So guess (iii) false and this part true.]]

-	(1)	$(x \,.\, x) = (x \,.\, x)$	II
-	(2)	$\exists y\ (x \,.\, y) = (y \,.\, x)$	EI, 1
-	(3)	$\forall x\ \exists y\ (x \,.\, y) = (y \,.\, x)$	UI, 2

As there are no assumptions then the sentence is a theorem of Q.

(iii)

In $\mathcal{N}^{**}$ let x = α. If (x.y) = (y.x) then the only solution is y = α.
In $\mathcal{N}^{**}$ let x = β. If (x.y) = (y.x) then the only solution is y = β.

Therefore $\exists y\ \forall x\ (x.y) = (y.x)$ is not true in $\mathcal{N}^{**}$.

All the axioms of Q hold in $\mathcal{N}^{**}$. As $\exists y\ \forall x\ (x.y) = (y.x)$ does not hold in the interpretation $\mathcal{N}^{**}$ then, it follows by the Correctness Theorem, the sentence is not a theorem of Q.

Question 16

(i) **True**

Let A be the set of the Gödel numbers of the 7 axioms of Q.

Given any number n then we can write an algorithm to compare this number with the 7 numbers in our set. If there is a match we return 1 otherwise return 0.

Therefore, by Church's Thesis, the characteristic function of the set A is recursive, and so the set A is recursive. Therefore Q is recursively axiomatizable.

(ii) **True**

Assume that Q is not consistent. Therefore, by definition, there is a sentence Φ of the formal language such that both $\vdash_Q \Phi$ and $\vdash_Q \neg \Phi$.

The Correctness Theorem tells us that if a formula is derivable from Q that it is a logical consequence of Q providing that there is an interpretation of Q.

Since $\mathcal{N}$ is an interpretation of Q then both Φ and ¬ Φ are logical consequences of Q. Since Φ cannot be both true and false in an interpretation then our assumption that Q is not consistent must be incorrect. So Q is consistent and the statement is true.

(iii) **False**

By Unit 7, Theorem 2.1(b) $\vdash_Q \mathbf{0} \neq \mathbf{2}$. So $\vdash_Q \neg\, \mathbf{0} = \mathbf{2}$.

Since Q is consistent, by part (ii), then both $\vdash_Q \mathbf{0} = \mathbf{2}$ and $\vdash_Q \neg\, \mathbf{0} = \mathbf{2}$ cannot be true.

Therefore $\vdash_Q \mathbf{0} = \mathbf{2}$ is not true.

OR

1	(1)	$\forall x \neg\, \mathbf{0} = x'$	Ass
1	(2)	$\neg\, \mathbf{0} = \mathbf{0}''$	UE, 1

As the assumption is an axiom of Q then $\vdash_Q \neg\, \mathbf{0} = \mathbf{2}$.

Since Q is consistent ….

(iv) **False**

Gödel's First Completeness theorem states that there is no complete and consistent recursively axiomatizable extension of Q.

Since Q is an extension of Q and, as we have shown in parts (i) and (ii), it is consistent and recursively axiomatizable this means Q cannot be complete.

[[See 2005 Qu 16 (ii)(b) for an alternative solution]]

END OF PART II SOLUTIONS