Question 9

(i)(a) (2 marks)

Next Instruction	R_1	R_2	R_3	R_4
1	2	3	1	0
2	2	3	1	0
3	2	4	1	0
4	2	4	1	1
1	2	4	1	1
5	2	4	1	1
STOP	4	4	1	1

(i)(b) (5 marks)

$f_P^1 : \mathbb{N} \to \mathbb{N}$ is defined by $f_P^1(n) = 0$. It is a total function.

$f_P^2 : \mathbb{N}^2 \to \mathbb{N}$ is defined by $f_P^2(n, m) = m$. It is a total function.

$f_P^3 : \mathbb{N}^3 \to \mathbb{N}$ is defined by $f_P^3(n, m, p) = m + p$. It is a total function.

$f_P^4 : \mathbb{N}^4 \to \mathbb{N}$ is defined by $f_P^4(n, m, p, q) = \begin{cases} m + p - q, & \text{if } p \geq q \\ \text{undefined}, & \text{otherwise.} \end{cases}$

(ii) (4 marks)

1 S(2)
2 S(2)
3 J(1, 2, 6)
4 Z(1)
5 S(1).

Question 10

(i) (2½ marks) [[See Unit 2, Problem 1.9]]

Let $\chi_{\leq}: \mathbb{N}^2 \to \mathbb{N}$ be defined by $\chi_{\leq}(n,m) = \overline{sg}\,(n \dot{-} m)$.

If n > m then $n \dot{-} m > 0$ and so $\chi_{\leq}(n, m) = 0$.
If $n \leq m$ then $n \dot{-} m = 0$ and so $\chi_{\leq}(n, m) = 1$.

Since $\chi_{\leq}$ is obtained by substitution from the total primitive recursive functions $\overline{sg}$ and $\dot{-}$ then $\chi_{\leq}$ is also a total primitive recursive function.

Therefore $\chi_{\leq}$ is a characteristic function for the relation $\leq$ and so it is primitive recursive.

(ii) 2½ marks [[See Unit 2, Problem 1.10]]

The characteristic function χ_R of the relation R is primitive recursive.

Let $\chi_T(n_1, n_2, ..., n_k) = \overline{sg}(\chi_R(n_1, n_2, ..., n_k))$.

Therefore the relation T is primitive recursive since its characteristic function is obtained by substitution from the primitive functions $\overline{sg}$ and χ_R.

(iii) 6 marks [[Use of Unit 2 Theorem 1.5]]

Define the functions

$g_1(m, n) = 7 = C_7^2\,(m, n)$,
$g_2(m, n) = mn = \text{mult}(m, n)$,
$g_3(m, n) = n^3 = \exp(n, 3)$

and the relations

$R_1(n, m) \Leftrightarrow 60 \leq \min(n, m)$,
$R_2(n, m) \Leftrightarrow 4n + 3m = 400$,
$R_3(n, m) \Leftrightarrow$ not $R_1(n, m)$ and not $R_2(n, m)$.

Then we can write

$$f(n, m) = \begin{cases} g_1(m,n) & \text{if } R_1(m,n) \\ g_2(m,n) & \text{if } R_2(m,n) \\ g_3(m,n) & \text{if } R_3(m,n) \end{cases}$$

As g_1, g_2, and g_3 can be written the primitive recursive functions C_7^2, mult, and exp using constants then g_1, g_2, and g_3 are primitive recursive functions.

The characteristic function of the relation R_1, $\chi_{R_1}(n,m) = \chi_{\leq}(60, \min(n,m))$. As χ_{R_1} is obtained by substitution from the primitive recursive functions $\chi_{\leq}$, and min using constants, then it is a primitive recursive function. Hence R_1 is a primitive recursive relation.

The characteristic function of the relation R_2, $\chi_{R2}(n,m) = \chi_{eq}(4n + 3m, 400)$. As χ_{R_2} is obtained by substitution from the primitive recursive functions χ_{eq}, mult and add using constants, then it is a primitive recursive function. Hence R_2 is a primitive recursive relation.

Using the result of Unit 2 Problem 1.10, then R_3 is also a primitive recursive relation.

From the definition of R_3 it follows that the set of relations R_1, R_2, and R_3 are exhaustive.

If the relation R_1 holds then $m \geq 60$ and $n \geq 60$. Therefore $4n + 3m \geq 4 * 60 + 3 * 60 = 420$. Since R_2 does not hold then R_1 and R_2 are mutually exclusive. From the definition of R_3, if the relation R_3 holds then neither R_1 or R_2 holds. Therefore R_1, R_2 and R_3 are mutually exclusive.

Since all the conditions required for the use of Theorem 1.5 of Unit 2 hold then it follows that f is primitive recursive.

Question 11

(i)(a) (2 marks)

Let $C_2^3(n_1, n_2, n_3) = \text{succ}(\text{succ}(\text{zero}(U_1^3(n_1, n_2, n_3))))$.

As C_2^3 is obtained by substitution from the basic primitive recursive functions succ, zero, and U_1^3 then C_2^3 is a primitive recursive function.

(i)(b) (3 marks)

Let mult(n, 0) = zero(n),
and mult(n, m + 1) = g(n, m, mult(n, m))
where $g(n_1, n_2, n_3) = \text{add}(U_1^3(n_1, n_2, n_3), U_3^3(n_1, n_2, n_3))$.

g is a primitive recursive function since it is obtained by substitution from add and the basic primitive recursive functions U_1^3, and U_3^3. Since zero is a basic primitive recursive functions and g is a primitive recursive function then mult is a primitive recursive function.

(ii)(a) (2 marks)

Let R be the relation 'the remainder of y on division by 4 is 1'. A characteristic function for R is,

$$\chi_R(y) = \chi_{eq}(\text{rem}(y,4), 1).$$

As χ_R is obtained by substitution from the primitive recursive functions χ_{eq} and rem using constants then χ_R is a primitive recursive function. Therefore R is a primitive recursive relation.

(ii)(b) 4 marks

An improved version by Lisette Petrie.

$f(n) = \mu y(n < y$ and the remainder of y on division by 4 is 1)
Consider the relation T given by
$T(n, y) \Leftrightarrow n < y$ and the remainder of y on division by 4 is 1.
The relation < is primitive recursive [*HB p21*] and the relation “the remainder of y on division by 4 is 1” is primitive recursive by part (a).
Then $\chi_T(n, y) = \chi_<(n, y)$ and $\chi_R(y)$,
which is obtained by substitution from the primitive recursive functions $\chi_<$ and χ_R and so is primitive recursive [*HB p21 result of problem 1.10*].
By Theorem 3.5 *[HB p23]* the function $g : \mathbb{N}^2 \to \mathbb{N}$ given by

$$g(n, z) = \mu y \leq z\ T(n, y)$$

is primitive recursive.
A suitable bound on y in terms of n is $n + 4$, so

$$f(n) = \mu y \leq (n + 4)\ T(n, y) = g(n, n + 4)$$

which is a substitution of primitive recursive functions using constants, so is primitive recursive.

Question 12

Solution by Wilson Stother.

(i) (2 marks)

For any natural number n, define the URM program I(n) by
(1) C(1,n).
This implements the identity function *id* since it halts with the original value in register 1.
Thus there are infinitely many such programs.

(ii) (5 marks)

Given a program P, first determine the maximum register number ρ(P) used by P.
Now create P* by concatenating the following programs
(1) C(1, ρ(P)+1),
P, and
(1) C(ρ(P)+1,1).

If f_P^1 is total, then P* saves the input in a register not used by P, executes P, and, as f_P^1 is total, moves on to the last instruction of P* which restores the original value of register 1. Thus P* computes the identity function *id* in this case.

If f_P^1 is not total, then P* will not halt for some input n. For this input, P* will execute the first instruction. This does not affect register 1, so the program P will not halt. Thus, P* will *not* halt for this input. Hence the function computed by P* is not total, and so P* cannot compute the identity function.

(iii) (4 marks)

Suppose that X is recursive. Then there is an algorithm for testing whether a code number e* is in X.

Tot is the set of code numbers of URMs which compute total functions of one variable.
We now have the following algorithm for deciding whether a given integer e is in *Tot*.

> There is a simple algorithm to check whether e is the code number of a URM.
> If not, then e is not in *Tot*.
> Otherwise, we can recover P from e, and construct P* as in (ii).
> Let P* have code number e*.
> From part (ii), $e \in Tot$ if and only if $e^* \in X$.

The existence of such an algorithm contradicts Theorem 3.2, so our assumption must be false.

Hence X is not recursive.

Question 13

(i) (3 marks)

Let θ be the formula $x' = y$; φ be $\exists x\, x' = y$; and ψ be $\forall x\, (x' = y \vee \exists x\, x' = y)$.

The given formula can then be written as $(((\theta \vee \varphi) \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\theta))$.

A truth table for this formula is

θ	ϕ	ψ	$(((\theta \vee \phi)\rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\theta))$
1	1	1	1 1 1 1 1 1 01 1 01
1	1	0	1 1 1 0 0 1 10 0 01
1	0	1	1 1 0 1 1 1 01 1 01
1	0	0	1 1 0 0 0 1 10 0 01
0	1	1	0 1 1 1 1 1 01 1 10
0	1	0	0 1 1 0 0 1 10 1 10
0	0	1	0 0 0 1 1 1 01 1 10
0	0	0	0 0 0 1 0 1 10 1 10
			(2) (3) (4) (2) (3) (2)

Since column 4 is all ones then the formula takes the truth value 1 under all interpretations.

(ii)(a) (2½ marks)

(a) 2 1/2 marks. *Same as 2003 with lines 1 and 2 interchanged.*

Line	1	2	3	4	5	6	7	8	9
Ass.	1	2	1	4	1,2	1,2	1,4	1	1,4

(ii)(b) (½ mark)

$(((\theta \rightarrow \psi) \,\&\, (\psi \,\&\, \phi)) \rightarrow (\theta \rightarrow \psi))$

(ii)(c) (2 marks)

(A) NO (Correction by Linda Brown). (B) YES.

(iii) 3 marks. This is the solution given to Problem 2.8 in Unit 5.

Let ψ be the formula $v = 0$, and ϕ be the same formula $v = 0$. ϕ is trivially a logical consequence of ψ and v occurs freely in ψ.

In the standard interpretation $\mathcal{N}$ give v the value 0 in the domain $\mathbb{N}$. Then the formula $v = 0$ is true, but $\forall v\, v = 0$ derived by use of the UI rule is false.

Question 14

(i) (2 marks)

$\forall \mathbf{z}\,(\forall x \exists t\; t = (y.x) \leftrightarrow \forall \mathbf{y}(y' + z) = (\mathbf{x} + t))$

(a) NO. [[y becomes bound]] (b) NO. [[z becomes bound]] (c) YES

(ii)(a) (3 marks)

1	(1)	$\exists y\, \forall x\, (x + y) = y$	Ass
2	(2)	$\forall x\, (x + y) = y$	Ass
2	(3)	$(x' + y) = y$	UE, 2
2	(4)	$\exists y\, (x' + y) = y$	EI, 3
2	(5)	$\exists x\, \exists y\, (x' + y) = y$	EI, 4
1	(6)	$\exists x\, \exists y\, (x' + y) = y$	EH, 5

Therefore $\exists y\, \forall x\, (x + y) = y \vdash \exists x\, \exists y\, (x' + y) = y$.

(ii)(b) (6 marks)

1	(1)	ϕ	Ass
2	(2)	$\forall x(\theta \vee \neg\phi)$	Ass
3	(3)	$\exists x\neg(\psi \to \theta)$	Ass. Contradiction
4	(4)	$\neg(\psi \to \theta)$	Ass
4	(5)	$\neg\theta$	Taut, 4
2	(6)	$(\theta \vee \neg\phi)$	UE, 2
2,4	(7)	$\neg\phi$	Taut, 5, 6
1,2,4	(8)	$(\phi \,\&\, \neg\phi)$	Taut, 1, 7
1,2,3	(9)	$(\phi \,\&\, \neg\phi)$	EH, 8
1,2	(10)	$(\exists x\neg(\psi \to \theta) \to (\phi \,\&\, \neg\phi))$	CP, 9
1,2	(11)	$\neg\exists x\neg(\psi \to \theta)$	Taut, 10
1	(12)	$(\forall x(\theta \vee \neg\phi) \to \neg\exists x\neg(\psi \to \theta))$	CP, 11

The assumption that x does not occur free in ϕ is required for the use of EH on line (9).

Question 15 (11 marks)

(i) [[Both sides of equation look as if they equal $(\mathbf{0} + x)$.]]

-	(1)	$((\mathbf{0} + \mathbf{0}) + x) = ((\mathbf{0} + \mathbf{0}) + x)$	II
2	(2)	$\forall x\ (x + \mathbf{0}) = x$	Ass. Q4
2	(3)	$(\mathbf{0} + \mathbf{0}) = \mathbf{0}$	UE, 2
2	(4)	$((\mathbf{0} + \mathbf{0}) + x) = (\mathbf{0} + x)$	Sub, 1, 3
-	(5)	$(\mathbf{0} + (x + \mathbf{0})) = (\mathbf{0} + (x + \mathbf{0}))$	II
2	(6)	$(x + \mathbf{0}) = \mathbf{x}$	UE, 2
2	(7)	$(\mathbf{0} + x) = (\mathbf{0} + (x + \mathbf{0}))$	Sub, 5, 6
2	(8)	$((\mathbf{0} + \mathbf{0}) + x) = (\mathbf{0} + (x + \mathbf{0}))$	Sub, 4, 7
2	(9)	$\forall x\ ((\mathbf{0} + \mathbf{0}) + x) = (\mathbf{0} + (x + \mathbf{0}))$	UI, 8

As the assumption is an axiom of Q then the sentence is a theorem of Q.

(ii) [[If this is a theorem then so is (iii). Therefore probably not one.]]

In $\mathcal{N}^{**}$ let $x = \alpha$, and $y = \alpha$. Then

$$((x + y) + x) = ((\alpha + \alpha) + \alpha) = (\beta + \alpha) = \beta, \text{ and}$$
$$(x + (y + x)) = (\alpha + (\alpha + \alpha)) = (\alpha + \beta) = \alpha.$$

All the axioms of Q hold in $\mathcal{N}^{**}$. As $\forall x \forall y\ ((x + y) + x) = (x + (y + x))$ does not hold in $\mathcal{N}^{**}$ then, it follows by the Correctness Theorem, the sentence is not a theorem of Q.

(iii) Solution by Wilson Stother.

1	(1)	$\forall x\ (x + \mathbf{0}) = x$	Ass. Q4
1	(2)	$((\mathbf{0} + y) + \mathbf{0}) = (\mathbf{0} + y)$	UE, 1
-	(3)	$(\mathbf{0} + (y + \mathbf{0})) = (\mathbf{0} + (y + \mathbf{0}))$	II
1	(4)	$(y + \mathbf{0}) = y$	UE, 1
1	(5)	$(\mathbf{0} + y) = (\mathbf{0} + (y + \mathbf{0}))$	Sub, 3, 4
1	(6)	$((\mathbf{0} + y) + \mathbf{0}) = (\mathbf{0} + (y + \mathbf{0}))$	Sub, 2, 5
1	(7)	$\forall y\ ((\mathbf{0} + y) + \mathbf{0}) = (\mathbf{0} + (y + \mathbf{0}))$	UI, 6
1	(8)	$\exists x\ \forall y\ ((\mathbf{x} + y) + \mathbf{x}) = (\mathbf{x} + (y + \mathbf{x}))$	EI, 7

As the assumption is an axiom of Q then the sentence is a theorem of Q.

Question 16

Solution by Linda Brown.

(i) (3 marks)

Suppose that theory T is not consistent but has an interpretation

Hence there is a sentence of T, Φ say, such that $\vdash_T \Phi$ and $\vdash_T \neg \Phi$,
i.e. both Φ and ¬ Φ are theorems of T

By the Correctness Theorem both Φ and ¬ Φ are true in every interpretation in which the sentences of T are true and so must be true in the interpretation of T

However a sentence cannot be both true and false in the same interpretation and this contradicts our original supposition

Hence a theory which has an interpretation is consistent

(ii)(a) (3 marks)

Theory Q is consistent as a consequence of Unit 6, Problem 3.4
Hence ¬ 0 = 0' and 0 = 0' cannot both be theorems of Q

∀x ¬ 0 = x' is an axiom of Q
Hence for x = 0, ¬ 0 = 0', i.e. 0 ≠ 1
Therefore ¬ 0 = 0' is a theorem of Q

Hence 0 = 0', i.e. 0 = 1 cannot be a theorem of Q

(ii)(b) (1 mark)

∀x (0 + x) = x

(iii) (4 marks)

(iii)(a) Theory CA is consistent because it has an interpretation, $\mathcal{N}$, by part (i)

(iii)(b) As theory CA consists of all the sentences of the formal language that are true in the standard interpretation $\mathcal{N}$, i.e. for every sentence either $\vdash_{CA} \Phi$ or $\vdash_{CA} \neg \Phi$, therefore CA is complete.
CA is also consistent by part (ii)(a)
CA is an extension of Theory Q because the axioms of Q are derivable in CA

Hence by Theorem 2.3, Gödel's Incompleteness Theorem, CA is not recursively axiomatizable, i.e. CA does not have a recursive set of axioms.

[Alternatively Theorem 2.4 gives this result directly]