Question 10

(i) **(4 marks)** Similar to Unit 1, Example 3.3.

(a) h(3, 0) = f(3) = 4.

(b) h(3, 0 + 1) = g(3, 0, h(3, 0)) = exp (1, 5) = 1.
h(3, 1 + 1) = g(3, 1, h(3, 1)) = exp (2, 2) = 4.

(ii)(a) **(3 marks)** See Unit 2, Example 1.5

Let pred(0) = 0 and
pred(n + 1) = n.

pred is a primitive recursive function since it can be defined by primitive recursion in terms of the basic primitive recursive functions.

Let dif(x, 0) = x, and dif(x, y + 1) = g(x, y, dif(x, y)) where $g(n_1, n_2, n_3) = \text{pred}(n_3)$.
g is primitive recursive since it is defined by substitution from the primitive recursive function pred. Since dif is defined by primitive recursion from primitive recursive functions then dif is primitive recursive.

(ii)(b) **(2 marks)**

$\overline{\text{sg}}(x) = \text{dif}(1, x)$. Since $\overline{\text{sg}}$ is obtained by substitution from the primitive recursive function dif using constants then $\overline{\text{sg}}$ is a primitive recursive function.

$\text{sg}(x) = \text{dif}(1, \overline{\text{sg}}(x))$. Since sg is obtained by substitution from the primitive recursive functions dif and $\overline{\text{sg}}$ using constants then sg is a primitive recursive function.

(iii) **(2 marks)**

$(0, x_2)$ for any natural number x_2.
(x_1, x_2) where $x_2 \leq x_1$ and $x_1 > 0$. [[y = 0 in all cases]]

Question 11

(i) (3 marks)

Topic not covered in post-2003 course.

(ii)(a) (2 marks) See Unit 2, Example 1.8.

$\chi_{eq}(x, y) = \overline{sg}(adf(x, y))$.

Since χ_{eq} is defined by substitution using the primitive recursive functions $\overline{sg}$ and adf then it is also primitive recursive. Therefore eq is a primitive recursive relation.

(ii)(b) (2½ marks)

Define χ_{Odd} by

$\chi_{Odd}(0) = 0$

$\chi_{Odd}(n + 1) = \overline{sg}(\chi_{Odd}(n))$.

χ_{Odd} is primitive recursive since it is defined by primitive recursion from the constant 0 and the primitive recursive function $\overline{sg}$. Therefore Odd is a primitive recursive set.

(ii)(c) (3½ marks)

Define the functions

$g_1(x, y) = x^{2y} = \exp(x, 2y)$

$g_2(x, y) = y + 3$,

$g_3(x, y) = 2 = C_2^2(x, y)$,

and the relations

$R_1(x, y) \Leftrightarrow x.y + 3$ is even,

$R_2(x, y) \Leftrightarrow 3x + 4y = 8000$,

$R_3(x, y) \Leftrightarrow$ not $R_1(x, y)$ and not $R_2(x, y)$.

Then we can write

$$f(x, y) = \begin{cases} g_1(x,y) & \text{if } R_1(x,y) \\ g_2(x,y) & \text{if } R_2(x,y) \\ g_3(x,y) & \text{if } R_3(x,y) \end{cases}$$

As g_1, g_2, and g_3 can be written the primitive recursive functions C_2^2, add, mult, and exp using constants then g_1, g_2, and g_3 are primitive recursive functions.

The characteristic function of the relation R_1, $\chi_{R_1}(x, y) = \chi_E(xy + 3)$. As χ_{R_1} is obtained by substitution from the primitive recursive functions χ_E, mult and add using constants, then it is a primitive recursive function. Hence R_1 is a primitive recursive relation.

The characteristic function of the relation R_2, $\chi_{R2}(x, y) = \chi_{eq}(3x + 4y, 8000)$. As χ_{R_2} is obtained by substitution from the primitive recursive functions χ_{eq}, mult and add using constants, then it is a primitive recursive function. Hence R_2 is a primitive recursive relation.

Using the result of Unit 2 Problem 1.10, then R_3 is also a primitive recursive relation.

From the definition of R_3 it follows that the set of relations R_1, R_2, and R_3 are exhaustive.

If the relation R_1 holds then both x and y are odd.. Therefore 3x + 4y is odd and cannot equal 8000. Since R_2 does not hold then R_1 and R_2 are mutually exclusive. From the definition of R_3, if the relation R_3 holds then neither R_1 or R_2 holds. Therefore R_1, R_2 and R_3 are mutually exclusive.

Since all the conditions required for the use of Theorem 1.5 of Unit 2 hold, then it follows that f is primitive recursive.

Question 12

(i) (3 marks)

Let $c(x, y) = \overline{sg}\ (\exp(y, 3) \dot{-} x)$.

c is primitive recursive since it is is defined by substitution using the primitive recursive functions $\overline{sg}$, $\dot{-}$ and exp using constants.

(ii) (3 marks)

You must not use the result of Unit 2, Theorem 3.1 on Bounded Summation.

Define the function g by
$g(x, 0) = 0$
$g(x, v + 1) = \text{add}(f(x, v + 1), g(x, v))$.

Since g is defined by primitive recursion in terms of the primitive recursive functions add, f, linear using constants then g is primitive recursive.

(iii) (4 marks)

Define the function k by $k(x) = g(x, x)$ where

$$g(x, v) = \begin{cases} \sum_{z=1}^{v} c(x, z) & \text{if } v \geq 1 \\ 0 & \text{if } v = 0 \end{cases}$$

If $x = 0$ then $k(x) = g(0, 0) = 0$.

If x > then $k(x) = \sum_{z=1}^{x} c(x, z)$. Eventually we will have $z^3 \geq x$. For all the values of $z^3 \leq x$ one will be added to the sum.

Since c is a primitive recursive function of 2 variables then by part (ii) we know that g is also primitive recursive. Therefore, by Unit 2 Problem 1.4, k is also primitive recursive.

(iv) (1 mark)

Define the function c where

$$c(x, y) = \begin{cases} 1 & \text{if } y^4 \leq x^3 \\ 0 & \text{otherwise} \end{cases}.$$

c is clearly primitive recursive. Define the function j by $j(x) = g(x, x)$ where g is defined as in part (iii). It then follows that j is also primitive recursive.

QUESTION 13

(i) 3 marks.

Let θ be the sub-formula $\exists x\ x = y$;
ϕ be the sub-formula $\forall x\ (x = y \leftrightarrow \exists x\ x = y)$;
ψ be the sub-formula $\forall x\ x = y$.

The given formula can be written as $((\neg\theta\ \&\ \neg(\phi \to \psi)) \to (\theta \vee \neg\psi))$

θ	ϕ	ψ	$((\neg\theta\ \&\ \neg(\phi \to \psi) \to (\theta \vee \neg\psi))$
1	1	1	01 0 0 1 1 1 1 1 1 01
1	1	0	01 0 1 1 0 0 1 1 1 10
1	0	1	01 0 0 0 1 1 1 1 1 01
1	0	0	01 0 0 0 1 0 1 1 1 10
0	1	1	10 0 0 1 1 1 1 0 0 01
0	1	0	10 1 1 1 0 0 1 0 1 10
0	0	1	10 0 0 0 1 1 1 0 0 01
0	0	0	10 0 0 0 1 0 1 0 1 10
			(2) (4)(3) (2) (5) (3)(2)

Since column 5 is all ones then the formula takes the truth value 1 under all interpretations.

(ii)(a) 2 1/2 marks.

Line	1	2	3	4	5	6	7	8	9
Ass.	1	2	2	4	1,2	1,2	2,4	2	2,4

(ii)(b) 1/2 mark.

$(\exists x\ (\theta \to \psi) \to ((\theta \to \psi) \to \exists x\ (\theta \to \psi)))$

(ii)(c) 2 marks.

(A) **NO** (B) **NO**.

(iii) 3 marks

I found this part of the question hard. I would have to learn the examples given in Unit 5.
This solution is Additional Exercise 3.6 in Unit 5. As this is a harder exercise I am surprised to see it in an exam paper.

Let ϕ be the formula $\forall y\ v = y$ and let τ be the term y, which is not substitutable for v in ϕ. The formula $\phi(\tau/v)$ is then $\forall y\ y = y$, which is true in all interpretations.

But $\exists v\ \phi$ is the formula $\exists v\ \forall y\ v = y$ which is false in the standard interpretation $\mathcal{N}$.

QUESTION 14

(i) **2 marks.**

. $\forall \mathbf{t}\ (\exists x\ \forall z\ (x.t) = (y + z) \rightarrow \exists \mathbf{y}(y + \mathbf{x}) = y)$

(a) YES (b) NO [[y becomes bound]] (c) NO [[t becomes bound]]

(ii) (a) - 3 marks.

[[This is a special case of Unit 5, Section 3.2, Example 3.6.]]

1	(1)	$\exists y\ \exists x\ (x + x) = y$	Ass
2	(2)	$\exists x\ (x + x) = y$	Ass
3	(3)	$(x + x) = y$	Ass
3	(4)	$\exists y(x + x) = y$	EI, 3
3	(5)	$\exists x\ \exists y\ (x + x) = y$	EI, 4
2	(6)	$\exists x\ \exists y\ (x + x) = y$	EH, 5
1	(7)	$\exists x\ \exists y\ (x + x) = y$	EH, 6

(ii) (b) - 6 marks.

1	(1)	$(\phi\ \&\ \forall x(\phi \rightarrow \psi))$	Ass
2	(2)	$\forall x(\neg\psi \vee \theta)$	Ass
3	(3)	$\exists x\ \neg\theta$	Ass. Contradiction
4	(4)	$\neg\theta$	Ass
2	(5)	$(\neg\psi \vee \theta)$	UE, 2
2,4	(6)	$\neg\psi$	Taut, 4, 5
1	(7)	$\forall x(\phi \rightarrow \psi)$	Taut, 1
1	(8)	$\phi \rightarrow \psi$	UE, 7
1,2,4	(9)	$\neg\phi$	Taut, 6, 8
1	(10)	ϕ	Taut, 1
1,2,4	(11)	$(\phi\ \&\ \neg\phi)$	Taut, 9, 10
1,2,3	(12)	$(\phi\ \&\ \neg\phi)$	EH, 11
1,2	(13)	$(\exists x\ \neg\theta \rightarrow (\phi\ \&\ \neg\phi))$	CP, 12
1,2	(14)	$\neg\exists x\ \neg\theta$	Taut, 13
1	(15)	$(\forall x(\neg\psi \vee \theta) \rightarrow \neg\exists x\ \neg\theta)$	CP, 14

The assumption that x does not occur free in ϕ is required for the use of EH on line (12).

QUESTION 15

(i) [[Looks as if both sides of the equation equal $(\mathbf{0}.x)$.]]

-	(1)	$(\mathbf{0}.(x + \mathbf{0})) = (\mathbf{0}.(x + \mathbf{0}))$	II
2	(2)	$\forall x\ (x + \mathbf{0}) = x$	Ass. Q4
2	(3)	$(x + \mathbf{0}) = x$	UE, 2
2	(4)	$(\mathbf{0}.(x + \mathbf{0})) = (\mathbf{0}.x)$	Sub, 1, 3
-	(5)	$((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0})) = ((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0}))$	II
6	(6)	$\forall x\ (x.\mathbf{0}) = \mathbf{0}$	Ass. Q6
6	(7)	$(\mathbf{0}.\mathbf{0}) = \mathbf{0}$	UE, 6
6	(8)	$((\mathbf{0}.x) + \mathbf{0}) = ((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0}))$	Sub, 5, 7
2	(9)	$((\mathbf{0}.x) + \mathbf{0}) = (\mathbf{0}.x)$	UE, 2
2,6	(10)	$(\mathbf{0}.x) = ((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0}))$	Sub, 8, 9
2,6	(11)	$(\mathbf{0}.(x + \mathbf{0})) = ((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0}))$	Sub, 4, 10
2,6	(12)	$\forall x\ (\mathbf{0}.(x + \mathbf{0})) = ((\mathbf{0}.x) + (\mathbf{0}.\mathbf{0}))$	UI, 11

As the assumptions are axioms Q4 and Q6 of Q then the sentence is a theorem of Q.

(ii)

In $\mathcal{N}^{**}$ let $x = \alpha$. If $(x.y) = (y.x)$ then the only solution is $y = \alpha$.
In $\mathcal{N}^{**}$ let $x = \beta$. If $(x.y) = (y.x)$ then the only solution is $y = \beta$.

Therefore $\exists y\ \forall x\ (x.y) = (y.x)$ is not true in $\mathcal{N}^{**}$.

All the axioms of Q hold in $\mathcal{N}^{**}$. As $\exists y\ \forall x\ (x.y) = (y.x)$ does not hold in the interpretation $\mathcal{N}^{**}$ then, it follows by the Correctness Theorem, the sentence is not a theorem of Q.

(iii)

-	(1)	$(x.x) = (x.x)$	II
-	(2)	$\exists y\ (x.y) = (y.x)$	EI, 1
-	(3)	$\forall x \exists y\ (x.y) = (y.x)$	UI, 2

As there are no assumptions then the sentence is a theorem of Q.

END OF PART II SOLUTIONS