Question 10

(i) **(4 marks)** Similar to Unit 1, Example 3.3.

(a) h(3, 0) = f(3) = 5.

(b) h(3, 0 + 1) = g(3, 0, h(3, 0)) = mult (6, 1) = 6.
h(3, 1 + 1) = g(3, 1, h(3, 1)) = mult (7, 2) = 14.

(ii)(a) **(1½ marks)**

add(n, 0) = n.
add(n, m + 1) = g(n, m, add(n, m)) where $g(n_1, n_2, n_3) = succ(n_3)$.

g is primitive recursive since it is defined by substitution from the primitive recursive function succ and the projection function. Since sum is defined by primitive recursion from primitive recursive functions then sum is primitive recursive.

(ii)(b) **(3½ marks)**

mult(n, 0) = 0
mult(n, m + 1) = g(n, m, mult(n, m)) where $g(n_1, n_2, n_3) = add(n_1, n_3)$.

g is primitive recursive since it is defined by substitution from the primitive recursive function add. Since mult is defined by primitive recursion from primitive recursive functions then mult is primitive recursive.

$f(n_1, n_2, 0) = 1$
$f(n_1, n_2, m + 1) = h(n_1, n_2, m, f(n_1, n_2, m))$ where $h(n_1, n_2, n_3, n_4) = mult(add(n_2, n_1), n_4)$

h is primitive recursive since it is defined by substitution from the primitive recursive function mult and add. Since f is defined by primitive recursion from primitive recursive functions then f is primitive recursive.

(iii) **(2 marks)**

(x_1, x_2) where $x_1 \geq x_2$. [[y = 0]]
(x_1, x_2) where $x_2 \geq 9$ and $x_1 < x_2$. [[y > 0]]

Question 11

(i) (3 marks)

Topic not covered in post-2003 course.

(ii)(a) (2½ marks) [[See Unit 2, Problem 1.9.]]

$\chi_{\geq}(n, m) = \overline{sg}(m \dot{-} n)$.

Since $\chi_{\geq}$ is defined by substitution using the primitive recursive functions $\overline{sg}$ and $\dot{-}$ then it is also primitive recursive. Therefore $\geq$ is a primitive recursive relation.

(ii)(b) (2 marks) See Unit 2, Problem 1.5.

$\text{Max}(x, y) = \text{add}(x \dot{-} y, y)$.

Max is a primitive recursive function since it is defined by substitution using the primitive recursive functions add and $\dot{-}$.

(ii)(c) (3½ marks)

Define the functions

$g_1(x_1, x_2, x_3) = x_1 + 3x_2$,

$g_2(x_1, x_2, x_3) = 9 = C_9^3(x_1, x_2, x_3)$,

$g_2(x_1, x_2, x_3) = 3^{x_3} = \exp(3, x_3)$,

and the relations

$R_1(x_1, x_2, x_3) \Leftrightarrow x_1 + x_3 \geq \text{Max}(x_2, 30)$,

$R_2(x_1, x_2, x_3) \Leftrightarrow 5x_1 + 3x_2 + 4x_3 = 100$,

$R_3(x_1, x_2, x_3) \Leftrightarrow$ not $R_1(x_1, x_2, x_3)$ and not $R_2(x_1, x_2, x_3)$.

Then we can write

$$f(x_1, x_2, x_3) = \begin{cases} g_1(x_1, x_2, x_3) & \text{if } R_1(x_1, x_2, x_3) \\ g_2(x_1, x_2, x_3) & \text{if } R_2(x_1, x_2, x_3) \\ g_3(x_1, x_2, x_3) & \text{if } R_3(x_1, x_2, x_3) \end{cases}$$

As g_1, g_2, and g_3 can be written the primitive recursive functions add, mult, C_9^3, and exp, using constants then g_1, g_2, and g_3 are primitive recursive functions.

The characteristic function of the relation R_1, $\chi_{R_1}(x_1, x_2, x_3) = \chi_{\geq}(x_1 + x_3, \text{Max}(x_2, 30))$. As χ_{R_1} is obtained by substitution from the primitive recursive functions $\chi_{\geq}$, add and Max using constants, then it is a primitive recursive function. Hence R_1 is a primitive recursive relation.

The characteristic function of the relation R_2, $\chi_{R2}(x_1, x_2, x_3) = \chi_{eq}(5x_1 + 3x_2 + 4x_3, 100)$. As χ_{R_2} is obtained by substitution from the primitive recursive functions χ_{eq}, mult and add using constants, then it is a primitive recursive function. Hence R_2 is a primitive recursive relation.

Using the result of Unit 2 Problem 1.10, then R_3 is also a primitive recursive relation.

From the definition of R_3 it follows that the set of relations R_1, R_2, and R_3 are exhaustive.

If the relation R_1 holds then $x_1 + x_3 \geq 30$. Therefore as $5x_1 + 3x_2 + 4x_3 \geq 4(x_1 + x_3) \geq 120$ then R_2 does not hold. So R_1 and R_2 are mutually exclusive. From the definition of R_3, if the relation R_3 holds then neither R_1 or R_2 holds. Therefore R_1, R_2 and R_3 are mutually exclusive.

Since all the conditions required for the use of Theorem 1.5 of Unit 2 hold then it follows that f is primitive recursive.

Question 12

(i) (2½ marks)

Let c(x, y) = div (x, exp(2, y)).

c is primitive recursive since it is is defined by substitution using the primitive recursive functions div and exp using constants.

(ii) (3½ marks)

Define the function

$$h(x,v)=\begin{cases}\sum_{z=1}^{v} f(x,z) & \text{if } v \geq 1 \\ 0 & \text{if } v=0\end{cases}$$

Since f is primitive recursive then by the Bounded Summation Theorem (Unit 2, Th. 3.1) the function h is also primitive recursive.

Define the function g by

$$g(x, v, w) = h(x, w) \dot{-} h(x, v \dot{-} 1).$$

If $w < v$ then since $h(x, v \dot{-} 1) \geq h(x, w)$ then $g(x, v, w) = 0$.

If $w \geq v$ and $v \leq 1$ then $h(x, v \dot{-} 1) = 0$ so $g(x, v, w) = h(x, w)$.

If $w \geq v$ and $v > 1$ then $g(x, v, w) = \sum_{z=1}^{w} f(x,z) - \sum_{z=1}^{v-1} f(x,z) = \sum_{v}^{w} f(x,z)$.

As g is formed by substitution from the primitive recursive functions h and $\dot{-}$ then g is primitive recursive.

(iii) (5 marks)

Define the function k by k(x) = g(x, 1, x)
where g(x, v, w) is defined as above with f(x, z) replaced by c(x,z) (See part (i)).

If x = 0 then k(0) = 0. [[This is not defined in the question. A mistake?]].

If $x > 0$ then $k(x) = \sum_{z=1}^{x} c(x,z)$. Since $2^x > x$ then we will test all possible values of z where 2^z could be a divisor of x. For each power of 2^z which divides x then 1 is added to the sum.

Since c is a primitive recursive function of 2 variables then by part (ii) we know that g is also primitive recursive. Therefore, using the result of the Generalisation of Unit 2 Problem 1.4, k is also a primitive recursive function.

QUESTION 13

(i) **3 marks.**

Let θ be the sub-formula ∃x x = 0;
 ϕ be the sub-formula x = 0;
 ψ be the sub-formula ¬∀x (∃x x = 0 ↔ x = 0).

The given formula can be written as ((¬θ & (ϕ ∨ ¬ψ)) ∨ (ϕ → θ))

θ	ϕ	ψ	((¬θ & (ϕ ∨ ψ)) ∨ (ϕ → θ))
1	1	1	01 0 1 1 1 1 1 1 1
1	1	0	01 0 1 1 0 1 1 1 1
1	0	1	01 0 0 1 1 1 0 1 1
1	0	0	01 0 0 0 0 1 0 1 1
0	1	1	10 1 1 1 1 1 1 0 0
0	1	0	10 1 1 1 0 1 1 0 0
0	0	1	10 1 0 1 1 1 0 1 0
0	0	0	10 0 0 0 0 1 0 1 0
			(2) (3) (2) (4) (2)

Since column 4 is all ones then the formula takes the truth value 1 under all interpretations.

(ii)(a) **2 1/2 marks.**

Line	1	2	3	4	5	6	7	8	9
Ass.	1	2	1	2	5	-	1,2	1,2	1,5

(ii)(b) **1/2 mark.**

(((ψ → ¬θ) & (ψ ∨ ϕ)) → (ϕ ∨ ¬θ))

(ii)(c) **2 marks.**

(A) **NO** (B) **YES**.

(iii) **3 marks**

I found this part of the question hard. I would have to learn the examples given in Unit 5.
This solution is copied from Unit 6 Section 2.1.

Let ϕ be the formula ∃y y = (x + $0^{/}$), and τ_1 and τ_2 be the terms $0^{//}$ and (y + $0^{/}$) respectively.

$\tau_1 = \tau_2$ is the formula $0^{//}$ = (y + $0^{/}$).
ϕ(τ_1/x) is the formula ∃y y = ($0^{//}$ + $0^{/}$), and
ϕ(τ_2/x) is the formula ∃y y = ((y + $0^{/}$) + $0^{/}$).

The first 2 formulas are true in $\mathcal{N}$ but the 3rd formula is not.

QUESTION 14

(i) **2 marks.**

. $\exists \mathbf{z} (\exists \mathbf{t} (\mathbf{x} + y) = t \rightarrow \forall y \exists x (x + z) = y)$

(a) YES (b) NO [[z becomes bound]] (c) YES

(ii) (a) - 3 marks.

[[This is a special case of Unit 5, Section 3.2, Example 3.7.]]

1	(1)	$\exists x \forall y (y.x) = x$	Ass
2	(2)	$\forall y (y.x) = x$	Ass
2	(3)	$(y.x) = x$	UE, 2
2	(4)	$\exists x (y.x) = x$	EI, 3
2	(5)	$\forall y \exists x (y.x) = x$	UI, 4
1	(6)	$\forall y \exists x (y.x) = x$	EH, 5

(ii) (b) - 6 marks.

1	(1)	$\forall x(\phi \ \& \ \theta)$	Ass
2	(2)	ψ	Ass
3	(3)	$\exists x(\neg\phi \vee \neg\psi)$	Ass. Contradiction
4	(4)	$(\neg\phi \vee \neg\psi)$	Ass
1	(5)	$(\phi \ \& \ \theta)$	UE, 1
1,4	(6)	$\neg\psi$	Taut, 4, 5
1,2,4	(7)	$(\psi \ \& \ \neg\psi)$	Taut, 2, 6
1,2,3	(8)	$(\psi \ \& \ \neg\psi)$	EH, 7
1,2	(9)	$(\exists x(\neg\phi \vee \neg\psi) \rightarrow (\psi \ \& \ \neg\psi))$	CP, 8
1,2	(10)	$\neg\exists x(\neg\phi \vee \neg\psi)$	Taut, 9
1	(11)	$(\psi \rightarrow \neg\exists x(\neg\phi \vee \neg\psi))$	CP, 10

The assumption that x does not occur free in ψ is required for the use of EH on line (8). [[Note that assumption 2 also contains ψ.]]

QUESTION 15 - 11 marks

(i) [[Both sides of the equation look as if they equal ($\mathbf{0}$.x).]]

-	(1)	$((\mathbf{0} + \mathbf{0}).\, x) = ((\mathbf{0} + \mathbf{0}).\, x)$	II
2	(2)	$\forall x\, (x + \mathbf{0}) = x$	Ass. Q4
2	(3)	$(\mathbf{0} + \mathbf{0}) = \mathbf{0}$	UE, 2
2	(4)	$((\mathbf{0} + \mathbf{0}).x) = (\mathbf{0}.x)$	Sub, 1, 3
-	(5)	$((\mathbf{0}.\mathbf{0}).x) = ((\mathbf{0}.\mathbf{0}).x)$	II
6	(6)	$\forall x\, (x.\mathbf{0}) = \mathbf{0}$	Ass. Q6
6	(7)	$(\mathbf{0}.\mathbf{0}) = \mathbf{0}$	UE, 6
2,6	(8)	$(\mathbf{0}.x) = ((\mathbf{0}.\mathbf{0}).x)$	Sub, 5, 7
2,6	(9)	$((\mathbf{0} + \mathbf{0}).x) = ((\mathbf{0}.\mathbf{0}).x)$	Sub, 4, 8
2,6	(10)	$\forall x\, ((\mathbf{0} + \mathbf{0}).x) = ((\mathbf{0}.\mathbf{0}).x)$	UI, 9

As the assumptions are axioms Q4 and Q6 of Q then the sentence is a theorem of Q.

(ii) [[If this is a theorem then so is (iii). Therefore probably not one.]]

In $\mathcal{N}^{**}$ let x = α. Then

$((\mathbf{0}.x).\, \mathbf{0}) = ((\mathbf{0}.\, \alpha).\, \mathbf{0}) = (\alpha.\mathbf{0}) = \mathbf{0}$, and

$(\mathbf{0}.\, (\mathbf{0}.x)) = (\mathbf{0}.\, (\mathbf{0}.\, \alpha)) = (\mathbf{0}.\, \alpha) = \alpha$.

All the axioms of Q hold in $\mathcal{N}^{**}$. As $\forall x \forall y\, ((\mathbf{0}.x).\, \mathbf{0}) = (\mathbf{0}.\, (\mathbf{0}.x))$ does not hold in the interpretation $\mathcal{N}^{**}$ then, it follows by the Correctness Theorem, the sentence is not a theorem of Q.

(iii)

-	(1)	$((\mathbf{0}.\, \mathbf{0}).\, \mathbf{0}) = ((\mathbf{0}.\, \mathbf{0}).\, \mathbf{0})$	II
2	(2)	$\forall x\, (x.\, \mathbf{0}) = \mathbf{0}$	Ass. Q6
2	(3)	$(\mathbf{0}.\, \mathbf{0}) = \mathbf{0}$	UE, 2
2	(4)	$((\mathbf{0}.\, \mathbf{0}).\, \mathbf{0}) = (\mathbf{0}.\, \mathbf{0})$	Sub, 1, 3
-	(5)	$(\mathbf{0}.\, (\mathbf{0}.\, \mathbf{0})) = (\mathbf{0}.\, (\mathbf{0}.\, \mathbf{0}))$	II
2	(6)	$(\mathbf{0}.\, \mathbf{0}) = (\mathbf{0}.\, (\mathbf{0}.\, \mathbf{0}))$	Sub, 5, 3
2	(7)	$((\mathbf{0}.\, \mathbf{0}).\, \mathbf{0}) = (\mathbf{0}.\, (\mathbf{0}.\, \mathbf{0}))$	Sub, 4, 6
2	(8)	$\exists x\, ((\mathbf{0}.\, \mathbf{x}).\, \mathbf{0}) = (\mathbf{0}.\, (\mathbf{0}.\, \mathbf{x}))$	EI, 7

As the assumption is axiom Q6 of Q then the sentence is a theorem of Q.

END OF PART II SOLUTIONS