Question 10

(i) **(4 marks)** Similar to Unit 1, Example 3.3.

(a) h(3, 0) = f(3) = 4.

(b) h(3, 0 + 1) = g(3, 0, h(3, 0)) = add (5, 1) = 6.
h(3, 1 + 1) = g(3, 1, h(3, 1)) = add (7, 2) = 9.

(ii) (a) **(2 marks)**

See 2003 Qu. 10 (ii)(a) where it was worth 3 marks.

(ii) (b) **(2 marks)**

See 2003 Qu. 10 (ii)(b).

(iii) **(3 marks)**

(x_1, x_2) where $x_1 \leq x_2$. [[y = 0]]
(x_1, x_2) where $x_2 \leq 9$ and $x_1 > x_2$. [[y > 0]]

Question 11

(i) (3 marks)

Topic not covered in post-2003 course.

(ii)(a) (2½ marks)

See 2002 Qu. 11 (ii)(a).

(ii)(b) (2 marks) [[See Unit 2, Problem 1.6.]]

$\text{Min}(x, y) = x \dot{-} (x \dot{-} y)$.

Min is a primitive recursive function since it is defined by substitution using the primitive recursive function $\dot{-}$.

(ii)(c) (3½ marks)

Define the functions

$g_1(x_1, x_2, x_3) = 2x_2 = \text{mult}(2, x_2)$,

$g_2(x_1, x_2, x_3) = x_1^{x_3} = \exp(x_1, x_3)$,

$g_3(x_1, x_2, x_3) = 5 = C_5^3(x_1, x_2, x_3)$,

and the relations

$R_1(x_1, x_2, x_3) \Leftrightarrow \text{Min}(x_1, x_3) = 30 + x_2$,

$R_2(x_1, x_2, x_3) \Leftrightarrow 3x_1 + 2x_2 + x_3 \leq 100$,

$R_3(x_1, x_2, x_3) \Leftrightarrow$ not $R_1(x_1, x_2, x_3)$ and not $R_2(x_1, x_2, x_3)$.

Then we can write

$$f(x_1, x_2, x_3) = \begin{cases} g_1(x_1, x_2, x_3) & \text{if } R_1(x_1, x_2, x_3) \\ g_2(x_1, x_2, x_3) & \text{if } R_2(x_1, x_2, x_3) \\ g_3(x_1, x_2, x_3) & \text{if } R_3(x_1, x_2, x_3) \end{cases}$$

As g_1, g_2, and g_3 can be written the primitive recursive functions mult, exp and C_5^3, using constants then g_1, g_2, and g_3 are primitive recursive functions.

The characteristic function of the relation R_1, $\chi_{R_1}(x_1, x_2, x_3) = \chi_{eq}(\text{Min}(x_1, x_3), 30 + x_2)$. As χ_{R_1} is obtained by substitution from the primitive recursive functions χ_{eq}, Min and add using constants, then it is a primitive recursive function. Hence R_1 is a primitive recursive relation.

The characteristic function of the relation R_2, $\chi_{R2}(x_1, x_2, x_3) = \chi_{\le}(3x_1 + 2x_2 + x_3, 100)$. As χ_{R_2} is obtained by substitution from the primitive recursive functions $\chi_{\le}$, mult and add using constants, then it is a primitive recursive function. Hence R_2 is a primitive recursive relation.

Using the result of Unit 2 Problem 1.10, then R_3 is also a primitive recursive relation.
From the definition of R_3 it follows that the set of relations R_1, R_2, and R_3 are exhaustive.

If the relation R_1 holds then $x_1 \ge 30 + x_2$ and $x_3 \ge 30 + x_2$. Therefore $3x_1 + 2x_2 + x_3 \ge 120 + 6x_2$. Since $x_2 \ge 0$ then R_2 does not hold so R_1 and R_2 are mutually exclusive. From the definition of R_3, if the relation R_3 holds then neither R_1 or R_2 holds. Therefore R_1, R_2 and R_3 are mutually exclusive.

Since all the conditions required for the use of Theorem 1.5 of Unit 2 hold then it follows that f is primitive recursive.

Question 12

(i) (2½ marks)

Let $c(x, y) = \overline{sg}\,(\exp(2, y) \dot{-} x)$.

c is primitive recursive since it is defined by substitution using the primitive recursive functions $\overline{sg}$, $\dot{-}$ and exp using constants.

(ii) (3 marks)

See 2003 Qu. 12 (ii)

(iii) (4 marks)

Define the function lo by lo(x) = g(x, x) where

$$g(x, v) = \begin{cases} \sum_{z=1}^{v} c(x, z) & \text{if } v \geq 1 \\ 0 & \text{if } v = 0 \end{cases}$$

lo(0) = g(0, 0) = 0 as required.

If x > 0 then lo(x) = g(x, x) = $\sum_{z=1}^{x} c(x, z)$. Since $2^x > x$ then we must eventually come to a value of z where $2^z > x$. As 1 is added to the sum for each value of z where $2^z \leq x$ then the sum will be the value required.

Since c is a primitive recursive function of 2 variables then by part (ii) we know that g is also primitive recursive. Therefore, using the result of Unit 2 Problem 1.4, lo is also primitive recursive.

(iv) (1½ marks)

k(x) = exp(2, lo(x)).

k is primitive recursive since it is defined by substitution using the primitive recursive functions lo and exp using constants.

QUESTION 13

(i) **(3 marks)**

Let θ be the sub-formula x = 0;
ϕ be the sub-formula ∀x x = 0;
ψ be the sub-formula ∃x (x = 0 ∨ ∀x x = 0).

The given formula can be written as (((θ ∨ ϕ)→ ψ) → (¬ψ → ¬θ))

θ	ϕ	ψ	(((θ ∨ ϕ)→ ψ) → (¬ψ → ¬θ))
1	1	1	1 1 1 1 1 1 01 1 01
1	1	0	1 1 1 0 0 1 10 0 01
1	0	1	1 1 0 1 1 1 01 1 01
1	0	0	1 1 0 0 0 1 10 0 01
0	1	1	0 1 1 1 1 1 01 1 10
0	1	0	0 1 1 0 0 1 10 1 10
0	0	1	0 0 0 1 1 1 01 1 10
0	0	0	0 0 0 1 0 1 10 1 10
			(2) (3) (4) (2) (3) (2)

Since column 4 is all ones then the formula takes the truth value 1 under all interpretations.

(ii)(b) **(2½ marks)**

Line	1	2	3	4	5	6	7	8	9
Ass.	1	2	1	4	1,2	1,2	1,4	1	1,4

(ii)(b) **(½ mark)**

(((ϕ → θ) & (θ & ψ)) → (ϕ → θ))

(ii)(c) **(2 marks)**

(A) **NO** (B) **YES**.

(iii) **3 marks**

This solution has been copied from Unit 5 Section 3.2.

1	(1)	∃v v = 0′	Ass
2	(2)	v = 0′	Ass
1	(3)	v = 0′	EH, 2
1	(4)	∀v v = 0′	UI, 3

QUESTION 14

[[Note that - is used instead of ¬ in papers prior to 2004.]]

(i) (2 marks)

$\exists \mathbf{z} \, (\forall x \exists y \, (x + t) = z \; \& \; \forall \mathbf{t}(\mathbf{x}.t) = y)$

(a) NO. [[z becomes bound]] (b) NO. [[t becomes bound]] (c) YES

(ii) (a) (3 marks)

-	(1)	$(x.x) = (x.x)$	II
2	(2)	$x = y$	Ass
2	(3)	$(x.y) = (y.x)$	Sub, 1, 2
-	(4)	$(x = y \rightarrow (x.y) = (y.x))$	CP, 3
-	(5)	$\forall y(x = y \rightarrow (x.y) = (y.x))$	UI, 4
-	(6)	$\forall x \forall y(x = y \rightarrow (x.y) = (y.x))$	UI, 5

(ii) (b) (6 marks)

1	(1)	$(\phi \; \& \; \forall x(\neg\phi \vee \psi))$	Ass
2	(2)	$\exists x \, \neg\theta$	Ass
3	(3)	$\neg\theta$	Ass
4	(4)	$\forall x(\psi \rightarrow \theta)$	Ass. Contradiction
4	(5)	$(\psi \rightarrow \theta)$	UE, 4
3,4	(6)	$\neg\psi$	Taut, 3, 5
1	(7)	$\forall x(\neg\phi \vee \psi)$	Taut, 1
1	(8)	$(\neg\phi \vee \psi)$	UE, 7
1,3,4	(9)	$\neg\phi$	Taut, 6, 8
1	(10)	ϕ	Taut, 1
1,3,4	(11)	$(\phi \; \& \; \neg\phi)$	Taut, 9, 10
1,2,4	(12)	$(\phi \; \& \; \neg\phi)$	EH, 11
1,2	(13)	$(\forall x(\psi \rightarrow \theta) \rightarrow (\phi \; \& \; \neg\phi))$	CP, 12
1,2	(14)	$\neg\forall x(\psi \rightarrow \theta)$	Taut, 13
1	(15)	$(\exists x \, \neg\theta \rightarrow \neg\forall x(\psi \rightarrow \theta))$	CP, 14

The assumption that x does not occur free in ϕ is required for the use of EH on line (12). [[Note that assumption 1 also contains ϕ.]]

QUESTION 15

(i) [[Both sides of the equation look as if they equal $(\mathbf{0} + x)$.]]

-	(1)	$(\mathbf{0} + (x + \mathbf{0})) = (\mathbf{0} + (x + \mathbf{0}))$	II
2	(2)	$\forall x\ (x + \mathbf{0}) = x$	Ass. Q4
2	(3)	$(x + \mathbf{0}) = x$	UE, 2
2	(4)	$(\mathbf{0} + (x + \mathbf{0})) = (\mathbf{0} + x)$	Sub, 1, 3
-	(5)	$((\mathbf{0} + \mathbf{0}) + x) = ((\mathbf{0} + \mathbf{0}) + x)$	II
2	(6)	$(\mathbf{0} + \mathbf{0}) = \mathbf{0}$	UE, 2
2	(7)	$(\mathbf{0} + x) = ((\mathbf{0} + \mathbf{0}) + x)$	Sub, 5, 6
2	(8)	$(\mathbf{0} + (x + \mathbf{0})) = ((\mathbf{0} + \mathbf{0}) + x)$	Sub, 4, 7
2	(9)	$\forall x\ (\mathbf{0} + (x + \mathbf{0})) = ((\mathbf{0} + \mathbf{0}) + x)$	UI, 8

As the assumption is axiom Q4 of Q then the sentence is a theorem of Q.

(ii) [[If this is a theorem then so is (iii). Therefore unlikely to be one.]]

In $\mathcal{N}^{**}$ let x = α, and y = α. Then

$$(x.(y.x)) = (\alpha.(\alpha.\alpha)) = (\alpha.\beta) = \beta, \text{ and}$$
$$((x.y).x) = ((\alpha.\alpha).\alpha) = (\beta.\alpha) = \alpha.$$

All the axioms of Q hold in $\mathcal{N}^{**}$. As $\forall x\forall y(x.(y.x)) = ((x.y).x)$ does not hold in the interpretation $\mathcal{N}^{**}$ then, it follows by the Correctness Theorem, the sentence is not a theorem of Q.

(iii)

-	(1)	$(\mathbf{0}.(y.\ \mathbf{0})) = (\mathbf{0}.(y.\ \mathbf{0}))$	II
2	(2)	$\forall x\ (x.\ \mathbf{0}) = \mathbf{0}$	Ass. Q6
2	(3)	$(y.\ \mathbf{0}) = \mathbf{0}$	UE, 2
2	(4)	$(\mathbf{0}.(y.\ \mathbf{0})) = (\mathbf{0}.\ \mathbf{0})$	Sub, 1, 3
2	(5)	$(\mathbf{0}.\ \mathbf{0}) = \mathbf{0}$	UE, 2
2	(6)	$(\mathbf{0}.(y.\ \mathbf{0})) = \mathbf{0}$	Sub, 4, 5
-	(7)	$((\mathbf{0}.y).\ \mathbf{0}) = ((\mathbf{0}.y).\ \mathbf{0})$	II
2	(8)	$((\mathbf{0}.y).\ \mathbf{0}) = \mathbf{0}$	UE, 2
2	(9)	$\mathbf{0} = ((\mathbf{0}.y).\ \mathbf{0})$	Sub, 7, 8
2	(10)	$(\mathbf{0}.(y.\ \mathbf{0})) = ((\mathbf{0}.y).\ \mathbf{0})$	Sub, 6, 9
2	(11)	$\forall y\ (\mathbf{0}.(y.\ \mathbf{0})) = ((\mathbf{0}.y).\ \mathbf{0})$	UI, 10
2	(12)	$\exists x\ \forall y\ (x.(y.\ x)) = ((x.y).\ x)$	EI, 11

As the assumption is axiom Q6 of Q then the sentence is a theorem of Q.

END OF PART II SOLUTIONS