Question 1

(a) $\exp\left(3+\frac{1}{4}\pi i\right)=e^3\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\frac{e^3}{\sqrt{2}}(1+i)$ (A2, Sect. 4, Para. 1)

(b) The principal argument of – 8 is π (A1, Sect. 2, Para. 7).
Therefore the principal cube root of -8 is (A1, Sect. 3, Para. 3)

$$\sqrt[3]{8}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=1+\sqrt{3}\,i$$

(c) $i^{1-2\mathrm{i}}=i\,i^{-2\mathrm{i}}=i\exp(-2\mathrm{i}\,\mathrm{Log}\,i)$. (A2, Sect. 5, Para. 3)

$\mathrm{Log}(\mathrm{i}) = \log_e|i| + iArg(i)$ (A2,Sect. 5, Para. 1)
$= \log_e 1 + i\pi/2 = i\pi/2$.

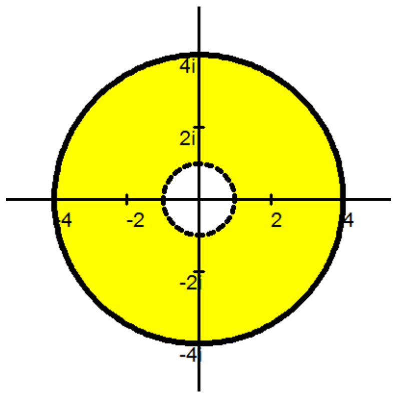
So $i^{1-2\mathrm{i}}=i\exp\left((-2\mathrm{i})\,i\,\frac{\pi}{2}\right)=i\,e^{\pi}$.

(d) $\cos(i\log_e 2)=\frac{1}{2}\left(\exp(i[i\log_e 2])+\exp(-i[i\log_e 2])\right)$ (A2, Sect. 4, Para. 4)

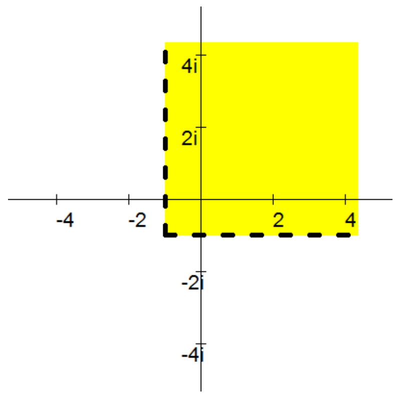
$$= \frac{1}{2}(\exp(-\log_e 2)+\exp(\log_e 2))=\frac{1}{2}\left(\frac{1}{2}+2\right)=\frac{5}{4}\ .$$

Question 2

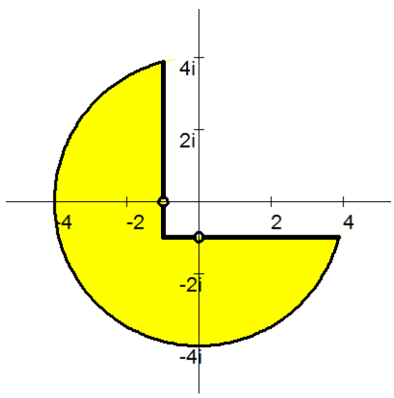
(a)



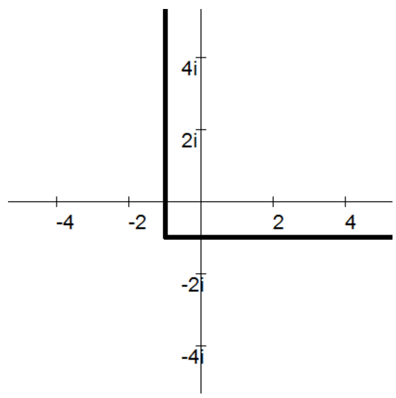
A



B



C = A - B



D = ∂B

[[The points -1 and – i are not in C.]]

(b)(i) A, C and D are not regions as they are not open. (A3, Sect. 4, Para.6)
B is a region.

(b)(ii) A and C are not compact as they are not closed (A3, Sect. 5, Para. 5)
B is not compact as it is neither closed nor bounded.
D is not compact as it is not bounded.

Question 3

Parts (a) and (b) same as parts 2007 (a) (i & ii).
(a)

The standard parametrization for the circle Γ is (Unit A2, Section 2, Para. 3)

$$\gamma(t) = 2(\cos t + i\sin t) = 2e^{it} \quad (t \in [0, 2\pi])$$

(b) $\gamma'(t) = 2ie^{it}$

Since γ is a smooth path then (Unit B1, Section 2, Para. 1)

$$\int_\Gamma \bar{z}\, dz = \int_0^{2\pi} \overline{\gamma(t)}\,\gamma'(t)dt = \int_0^{2\pi} 2e^{-it}(2ie^{it})\,dt = 4i\int_0^{2\pi} dt = 8\pi i$$

(b)

$f(z)=\dfrac{2\sin z}{\bar{z}^2+1}$ is continuous on $\mathbb{C}$ – {i, -i} and hence on the circle Γ. So we can use the Estimation Theorem (B1, Sect. 4, Para. 3) to obtain an upper estimate for the modulus of the integral.

The length of Γ is L = 2π*2 = 4π.

Using the Triangle Inequality (A1, Sect. 5, Para. 3b) then, for z = x +iy ∈ Γ, we have

$$|\sin z|=\frac{1}{2}|e^{iz}-e^{-iz}|\le\frac{1}{2}\{|e^{iz}|+|e^{-iz}|\}=\frac{1}{2}\{e^{\mathrm{Re}(iz)}+e^{\mathrm{Re}(-iz)}\} \quad \text{(A2, Sect. 4, Para. 2b)}$$

$$= \frac{1}{2}\{e^{-y}+e^{y}\}\le\frac{1}{2}\{e^{2}+e^{2}\}=e^{2}$$

Using the Backwards form of the Triangle Inequality (A1, Sect. 5, Para. 3c) then, for z ∈ Γ, we have

$$|\bar{z}^2+1|\ge||\bar{z}^2|-1|\ge|2^2-1|=3 \quad .$$

Therefore $M=\left|\dfrac{2\sin z}{\bar{z}^2+1}\right|\le\dfrac{2}{3}e^2$ for z ∈ Γ.

By the Estimation Theorem (B1, Sect. 4, Para. 3) $\int_C f(z)\,dz\le ML=\frac{2}{3}e^2 4\pi=\frac{8}{3}\pi e^2$

Question 4

Let ***R*** = {z: |z| < 2}.

(a)

R is a simply-connected region, $\dfrac{\operatorname{Log}(2-z)}{z^2+4}$ is analytic on ***R,*** and C is a closed contour in ***R***. So by Cauchy's Theorem (B2, Sect. 1, Para. 4)

$$\int_C \frac{\operatorname{Log}(2-z)}{z^2+4}\,dz = 0 \quad .$$

(b)

R is a simply-connected region, C is a simple-closed contour in ***R,*** $f(z)=\dfrac{\operatorname{Log}(2-z)}{z-2}$ is analytic on ***R,*** and 0 is inside C. So using Cauchy's Integral Formula (B2, Sect. 2, Para. 1) we have

$$\int_C \frac{\operatorname{Log}(2-z)}{z(z-2)}\,dz = \int_C \frac{f(z)}{z-0}\,dz = 2\pi i\, f(0) = 2\pi i\left(-\frac{\log 2}{2}\right) = -\pi i \log 2 \quad .$$

(c)

R is a simply-connected region, C is a simple-closed contour in ***R,*** $f(z)=\operatorname{Log}(2-z)$ is analytic on ***R,*** 0 is inside C and f(z) can be differentiated twice.

$$f^{(1)}(z) = \frac{-1}{2-z} \quad , \quad f^{(2)}(z) = \frac{-1}{(2-z)^2} \quad .$$

So using Cauchy's n'th Derivative Formula (B2, Sect. 3, Para. 1) we have

$$\int_C \frac{\operatorname{Log}(2-z)}{z^3}\,dz = \int_C \frac{f(z)}{(z-0)^3}\,dz = \frac{2\pi i}{2!} f^{(2)}(0) = \frac{-\pi i}{(2-0)^2} = -\frac{\pi i}{4} \quad .$$

Question 5

(a)

f has simple poles at z = 0, z = 1/5 and z = 5.

$$Res(f,0)=\lim_{z\to 0}(z-0)f(z)=\lim_{z\to 0}\frac{z^2+1}{(5z-1)(z-5)}=\frac{1}{5}$$ [[C1, Sect. 1, Para. 1]]

$$Res(f,1/5)=\lim_{z\to 1/5}(z-1/5)f(z)=\lim_{z\to 1/5}\frac{z^2+1}{z5(z-5)}=\frac{26}{25}\left(-\frac{5}{24}\right)=-\frac{13}{60}$$

$$Res(f,5)=\lim_{z\to 5}(z-5)f(z)=\lim_{z\to 5}\frac{z^2+1}{z(5z-1)}=\frac{26}{120}=\frac{13}{60}$$

(b)

I shall use the strategy given in C1, Sect. 2, Para. 2.

$$\int_0^{2\pi}\frac{\cos t}{13-5\cos t}dt=\int_C\frac{(z+z^{-1})/2}{13-5(z+z^{-1})/2}\frac{1}{iz}dz$$, where C is the unit circle {z : |z| = 1}

$$=\quad -i\int_C\frac{z^2+1}{z(26z-5z^2-5)}dz=i\int_C\frac{z^2+1}{z(5z-1)(z-5)}dz$$

The singularities of f(z) inside the unit circle C are at z = 0 and z = 1/5.

As f is a function which is analytic on the region $\mathbb{C}$ except for a finite number of singularities, and C is a simple closed contour in $\mathbb{C}$ which does not pass through any of the singularities then we can use Cauchy's Residue Theorem (C1, Sect. 2, Para. 1).

Therefore

$$\int_0^{2\pi}\frac{\cos t}{13-5\cos t}dt=i(2\pi i)\{Res(f,0)+Res(f,1/5)\}=-2\pi\left(\frac{1}{5}-\frac{13}{60}\right)=\frac{\pi}{30}$$

Question 6

(a)

In order to use Rouché's theorem we have to check that the 2 conditions listed in the Handbook (C2, Sect. 2, Para. 4) are satisfied.

The function f is analytic on the simply-connected region $\boldsymbol{R} = \mathbb{C}$ so condition 1 is satisfied.

(a)(i) Let $g_1(z) = iz^5$.

Using the Triangle Inequality (A1, Sect. 5, Para. 3) when $z \in C_1$ then

$$| f(z) - g_1(z) | = |5z^2 - 3i | \le |5z^2| + | -3i | = 20 + 3 < 2^5 = | g_1(z) |.$$

Since C_1 is a simple-closed contour in $\boldsymbol{R}$ and $| f(z) - g_1(z) | < | g_1(z) |$ on C_1 then the second condition is also satisfied. Therefore by Rouché's theorem f has the same number of zeros as g_1 inside the contour C_1. Therefore f has 5 zeros inside C_1.

(a)(ii) Let $g_2(z) = 5z^2$.

Using the Triangle Inequality when $z \in C_2$ we have

$$| f(z) - g_2(z) | = |iz^5 - 3i | \le |iz^5| + | -3i | = 1 + 3 < 5 = | g_2(z) |.$$

Since C_2 is a simple-closed contour in $\boldsymbol{R}$ and $| f(z) - g_2(z) | < | g_2(z) |$ on C_2 then the second condition is also satisfied. Therefore by Rouché's theorem f has the same number of zeros as g_2 inside the contour C_2. Therefore f has 2 zeros inside C_2.

(b)

From part(a) f(z) has 5 - 2 = 3 solutions in the set $\{z: 1 \le |z| < 2 \}$. Therefore we have to find if there are any solutions on C_2.

Since $|z_1 \pm z_2 \pm \ldots z_n| \ge |z_1| - |z_2| - \ldots -|z_n|$, (A1, Sect. 5, Para. 3(e))
then on C_2

$$|iz^5 + 5z^2 - 3i| = |5z^2 + iz^5 - 3i| \ge |5z^2| - |iz^5| - |3i| = 5 - 1 - 3 > 0.$$

As f(z) is non-zero on C_2 then there are exactly 3 solutions of f(z) = 0 in the set $\{z: 1 < |z| < 2 \}$.

Question 7

(a)

q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C}$ and the conjugate velocity function $\bar{q}(z)=z+1+i$ is analytic on $\mathbb{C}$. Therefore q is a model flow on $\mathbb{C}$ (Unit D2, Section 1, Para. 14).

(b)

The complex potential function Ω is given by

$$\Omega'(z) = \bar{q}(z)=z+1+i \quad . \qquad \text{(D2, Sect. 2, Para. 1)}$$

Therefore the complex potential function $\Omega(z)=\frac{z^2}{2}+(1+i)z$.

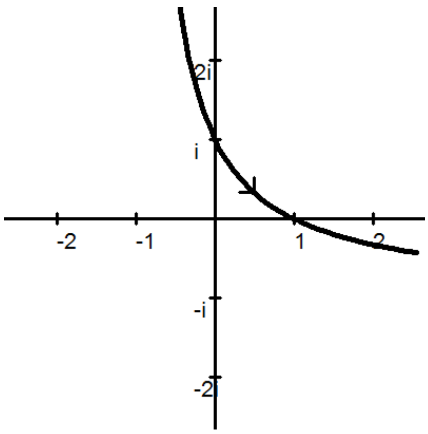
The stream function (Unit D2, Section 2, Para. 4)

$$\Psi(x,y)=\operatorname{Im}\Omega(z)=\operatorname{Im}\left(\frac{x^2-y^2+i2xy}{2}+(1+i)(x+iy)\right)=xy+x+y=(x+1)(y+1)-1$$

A streamline through 1 is given by $(x+1)(y+1)-1=\Psi(1,0)=2-1=1$.

So x and y are related by the equation $y=\frac{2}{x+1}-1$.

The direction of flow at 1 is given by the angle Arg q(1) = Arg (2 – i) which is downwards and to the right.



(c)

As q is a model flow velocity function on the region $\mathbb{C}$ and Γ lies in $\mathbb{C}$ then the circulation of q along Γ is, using the result given in D2, Sect. 2, Para. 1,

Re (Ω(β) – Ω(α)), where α and β are the start and end points of Γ.

As α = 0 and β = 4 then the required circulation is $\operatorname{Re}(8+4(1+i)-0)=12$.

Question 8

(a)

Using the result in Unit D3, Section 2, Para. 1 then the iteration sequence $z_{n+1} = 15z_n^2 + 3 z_n + 1/16$ is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + (15*1/16 + (3) / 2 - (3)^2 / 4) = w_n^2 + 3/16$$

and conjugating function $h(z) = 15z + 3/2$.

Therefore $w_0 = h(z_0) = 15z_0 + 3/2 = 3/2$. (Unit D3, Section 1, Para. 7).

(b)

If α is a fixed point of $P_{3/16}$ then $P_{3/16}(\alpha) = \alpha^2 + 3/16 = \alpha$ (D3, Sect. 1, Para 3).
As $\alpha^2 - \alpha + 3/16 = (\alpha - 3/4)(\alpha - 1/4) = 0$ then $P_{3/16}(z)$ has fixed points at $z = 3/4$ and $z = 1/4$.

$P_{3/16}'(z) = 2z$.

As $| P_{3/16}'(3/4)| = 3/2 > 1$ then this is a repelling fixed point (D3, Sect. 1, Para. 5).
As $| P_{3/16}'(1/4) | = 1/2 < 1$ then this an attracting fixed point (D3, Sect. 1, Para. 5).

(c)

$$c=-\frac{3}{2}+i$$

[[If you have added coordinates on the axes of the diagram of the Mandelbrot set then you will see that c is not in the Mandelbrot set.]]

$$P_c(0)=-\frac{3}{2}+i$$
$$P_c^2(0)=\left(-\frac{3}{2}+i\right)^2+\left(-\frac{3}{2}+i\right)=\left(\frac{9}{4}-1-\frac{3}{2}\right)+i(-3+1)=-\frac{1}{4}-2i$$

As $\left|P_c^2(0)\right|>2$ then c does not lie in the Mandelbrot set (D3, Sect. 4, Para. 5).

Question 9

(a)

(a)(i)

$$f(z)=z(3+\bar{z})+\operatorname{Re}z=3(x+iy)+(x^2+y^2)+x=u(x,y)+i\,v(x,y)$$
where $u(x,y) = 4x + x^2 + y^2$, and $v(x,y) = 3y$ are real valued functions.

(a)(ii)

f is defined on the region $\mathbb{C}$.

$$\frac{\partial u}{\partial x}=4+2x \quad , \quad \frac{\partial u}{\partial y}=2y \quad , \quad \frac{\partial v}{\partial x}=0 \quad , \quad \frac{\partial v}{\partial y}=3 \quad .$$

If f is differentiable then the Cauchy-Riemann equations hold (A4, Sect. 2, Para. 1). If they hold at (a, b) then

$$\frac{\partial u}{\partial x}(a,b)=4+2a=3=\frac{\partial v}{\partial y}(a,b)$$, and

$$\frac{\partial v}{\partial x}(a,b)=0=-2b=-\frac{\partial u}{\partial y}(a,b)$$

Therefore the Cauchy-Riemann equations only hold at (-1/2, 0).

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous at (-1/2, 0).
3. satisfy the Cauchy-Riemann equations at (-1/2, 0)

then, by the Cauchy-Riemann Converse Theorem (A4, Sect. 2, Para. 3), f is differentiable at -1/2.

As the Cauchy-Riemann equations only hold at (-1/2, 0) then f is not differentiable on any region surrounding -1/2. Therefore f is not analytic at -1/2. (A4, Sect. 1, Para. 3)

(a)(iii)

$$f'\left(-\frac{1}{2},0\right)=\frac{\partial u}{\partial x}\left(-\frac{1}{2},0\right)+i\frac{\partial v}{\partial x}\left(-\frac{1}{2},0\right)=3$$ (A4, Sect. 2, Para. 3).

(b)(i) g(z) is analytic on the region $\mathbb{C}-\{0\}$ (Unit A4, Section 3, Para. 4), and $g'(z)=1-\frac{i}{z^2}$ on $\mathbb{C}-\{0\}$. As $g'(1)=1-i$ and g is analytic at 1 , then g is conformal at z = 1. (Unit A4, Section 4, Para. 6)

(b)(ii) As g is analytic on $\mathbb{C}-\{0\}$ and $g'(1)\neq 0$ then a small disc centred at 1 is mapped approximately (Unit A4, Section 1, Para. 11) to a small disc centred at g(1) = 1 + i. The disc is rotated by Arg (g '(1)) = Arg 1 - i = -π/4, and scaled by a factor | g '(1)| = $2^{1/2}$.

(b)(iii) 0 is in the domain of γ_1 and $\gamma_1(0)=e^0=1$.

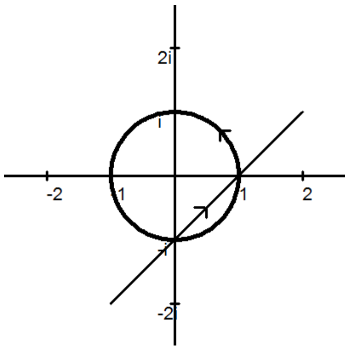
$\gamma_1'(t)=i e^{it}$ so $Arg\,\gamma_1'(0)=Arg(i e^0)=Arg\, i=\frac{\pi}{2}$.

1 is in the domain of γ_2 and $\gamma_2(1)=(1-1)i+1=1$.

$\gamma_2'(t)=1+i$ so $Arg\,\gamma_2'(0)=Arg(1+i)=\frac{\pi}{4}$

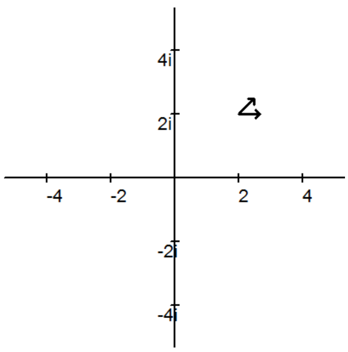
Therefore Γ_1 and Γ_2 meet at the point 1 and the angle from Γ_1 to Γ_2 is -π/4.

(b)(iv)



(b)(v)

In the diagram below g(Γ_2) is the horizontal line.



(b)(vi) $g(\Gamma_1)=e^{it}+i e^{-it}$ where $t\in[0,2\pi]$.

As $g'(e^{i\pi/4})=1-\frac{i}{e^{i\pi/2}}=1-\frac{i}{i}=0$ then g is not conformal at $e^{i\pi/4}$.

Question 10

(a)(i)

The singularities occur when the denominator of f is zero. Therefore there are singularities at z = 1 and z = 5. f has simple poles at z = 1 and z = 5 as $\lim_{z \to a} (z-a) f(z)$ exists when a = 1 or 5 (Unit B4, Section 3, Para. 2).

(a)(ii)

$$f(z)=\frac{1}{(z-1)(z-5)}=\frac{1}{4}\left(\frac{1}{z-5}-\frac{1}{z-1}\right)$$

$$\frac{1}{z-5}=\frac{1}{(z-2)-3}=\frac{1}{(-3)}\frac{1}{(1-(z-2)/3)}$$

On the annulus { z: 1 < | z – 2 | < 3}, | (z - 2) / 3 | < 1. Using the Basic Taylor Series for $(1 - z)^{-1}$ (Unit B3, Section 3, Para. 5) then

$$\frac{1}{z-5}=-\frac{1}{3}\left(1+\frac{z-2}{3}+\left(\frac{z-2}{3}\right)^2+\ldots\right) .$$

$$\frac{1}{z-1}=\frac{1}{(z-2)+1}=\frac{1}{z-2}\frac{1}{(1+1/(z-2))}$$

On the annulus { z: 1 < | z – 2 | < 3}, | 1 / (z - 2) | < 1. Using the Basic Taylor Series for $(1 - z)^{-1}$ (Unit B3, Section 3, Para. 5) then

$$\frac{1}{z-1}=\frac{1}{z-2}\left(1-\frac{1}{z-2}+\left(\frac{1}{z-2}\right)^2+\ldots\right)$$

So the required Laurent series about 2 for f on the annulus is

$$f(z)=\ldots-\frac{1}{4(z-2)}-\frac{1}{12}-\frac{z-2}{36}+\ldots$$

(b)(i)

The Taylor series for z sin z about 0 is $z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \ldots$. (B3, Sect. 3, Para. 5)

By the Composition Rule (Unit B3, Section 4, Para. 3) the Taylor series for g about 0 on $\mathbb{C}$ is

$$1 + \left(z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \ldots\right) + \frac{1}{2!}\left(z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \ldots\right)^2 + \frac{1}{3!}\left(z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \ldots\right)^3 + \ldots =$$

$$1 + z^2 + z^4\left(-\frac{1}{6} + \frac{1}{2}\right) + \ldots = 1 + z^2 + \frac{z^4}{3} + \ldots$$

As the series for exp and z sin z are valid for all $\mathbb{C}$ then the series represents g on $\mathbb{C}$.

(b)(ii)

z^3 g(1/z) is analytic on the punctured disc $\mathbb{C}$- {0}.

The Laurent series about 0 for z^3 g(1/z) on this disc is

$$z^3\left(1 + \frac{1}{z^2} + \frac{1}{3z^4} + \ldots\right) = z^3 + z + \frac{1}{3z} + \ldots$$

Therefore as C is a circle with centre 0 (Unit B4, Section 4, Para. 2)

$$\int_C z^3 g\left(\frac{1}{z}\right) dz = 2\pi i\, a_{-1} = \frac{2}{3}\pi i \quad ,$$

where a_{-1} is the coefficient of z^{-1} in z^3 g(1/z).

Question 11

(a)

$$f(z)=\frac{\pi\cot\pi z}{16\left(z-\frac{3}{4}i\right)\left(z+\frac{3}{4}i\right)}$$ has simple poles at $\pm\frac{3}{4}i$.

By the cover-up rule (Unit C1, Section 1, Para. 3)

$$Res\left(f,\frac{3i}{4}\right)=\frac{\pi\cot\left(\frac{3\pi i}{4}\right)}{16\left(\frac{3}{4}i+\frac{3}{4}i\right)}=\frac{\pi}{24i}\cot\left(\frac{3\pi i}{4}\right)$$

As sin(iz) = i sinh z and cos(iz) = cosh z then cot(iz) = - i coth(z). (Unit A2, Section 4, Para. 7).

Therefore $Res\left(f,\frac{3i}{4}\right)=-\frac{\pi}{24}\coth\left(\frac{3\pi}{4}\right)$

Similarly

$$Res\left(f,-\frac{3i}{4}\right)=\frac{\pi\cot\left(\frac{-3\pi i}{4}\right)}{16\left(\frac{-3}{4}i-\frac{3}{4}i\right)}=\frac{\pi}{-24i}\cot\left(\frac{-3\pi i}{4}\right)=\frac{\pi}{24}\coth\left(\frac{-3\pi}{4}\right)=-\frac{\pi}{24}\coth\left(\frac{3\pi}{4}\right)$$

(Unit A2, Section 4, Para. 6)

Let f(z) = g(z) / h(z) where $g(z)=\frac{\pi\cos\pi z}{16z^2+9}$ and h(z) = sin πz.

g and h are analytic at 0, h(0) = 0, and $h'(0) = \pi\cos(0) = \pi \neq 0$.

Therefore by the g/h rule (Unit C1, Section 1, Para. 2)

$$Res(f,0)=\frac{g(0)}{h'(0)}=\frac{\pi}{9}\frac{1}{\pi}=\frac{1}{9}\quad .$$

[You could also use Unit C1, Section 4, Para 1 – last line]

(b)

The method given in Unit C1, Section 4, Para. 1 will be used.

$f(z) = \pi \cot \pi z * \phi(z)$ where $\phi(z) = 1/(16z^2 + 9)$.

ϕ is an even function which is analytic on $\mathbb{C}$ except for simple poles at the non-integral points $z = \pm 3i/4$.

Let S_N be the square contour with vertices at $(N + ½)(\pm 1 \pm i)$.

On S_N we have $|z| \geq N + ½$ so, using the backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 2),

$$|16z^2 + 9| \geq | \, |16z^2| - 9 \, | \geq 16(N + ½)^2 - 9 \geq 16N^2.$$

On S_N we also have $\cot \pi z \leq 2$ (Unit C1, Section 4, Para. 2) so on S_N

$$| f(z) | \leq \frac{\pi (2)}{16N^2}$$

The length of the contour S_N is $4(2N + 1)$.

As f is continuous on the contour S_N then by the Estimation Theorem (Unit B1, Section 4, Para. 3) we have

$$\left| \int_{S_N} f(z)dz \right| \leq \frac{2\pi}{16N^2} 4(2N+1) = \frac{\pi}{2N}\left(2 + \tfrac{1}{N}\right)$$

Hence $\lim_{N \to \infty} \left| \int_{S_N} f(z)dz \right| = 0$.

Therefore the conditions specified in Unit C1, Section 4, Para. 1 hold so

$$\sum_{n=1}^{\infty} \frac{1}{16n^2+9} = -\frac{1}{2}\left(Res(f, 0) + Res\left(f, \frac{3i}{4}\right) + Res\left(f, -\frac{3i}{4}\right)\right)$$

$$= -\frac{1}{2}\left(\frac{1}{9} - \frac{\pi \coth(3\pi/4)}{24} - \frac{\pi \coth(3\pi/4)}{24}\right) = -\frac{1}{18} + \frac{\pi}{24}\coth\frac{3\pi}{4} \quad .$$

(c)

$$\sum_{-\infty}^{\infty} \frac{1}{16n^2+9} = \sum_{-\infty}^{-1} \frac{1}{16n^2+9} + \frac{1}{9} + \sum_{1}^{\infty} \frac{1}{16n^2+9} = \frac{1}{9} + 2\sum_{1}^{\infty} \frac{1}{16n^2+9} = \frac{\pi}{12}\coth 3\frac{\pi}{4} \quad .$$

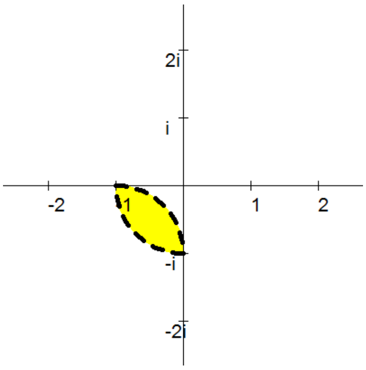
Question 12

(a)

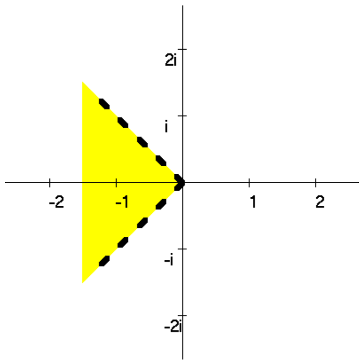
Using the formula for a transformation mapping points to the standard triple (D1, Sect. 2, Para. 11) then the Möbius transformation $\hat{f}_1$ which maps -1, ∞, and - i to the standard triple of points 0, 1, and ∞ respectively is

$$f_1(z)=\frac{(z-(-1))(\infty-(-i))}{(z-(-i))(\infty-(-1))}=\frac{z+1}{z+i}$$

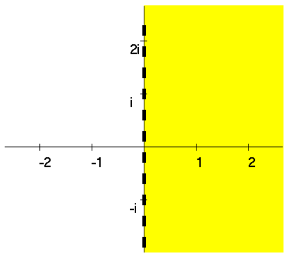
(b)(i)



R



S



T

(b)(ii) Since $\hat{f}_1$ maps -1 to 0 and -i to ∞ then the curved boundaries of R are mapped to extended lines originating at the origin. As the angle between the arcs of the circles meeting at -1 is $\pi/2$ and the mapping $\hat{f}_1$ is conformal then the angle between the extended lines at the origin is also $\pi/2$.

The standard parameterization of the line segment from -1 to -i is

$$z(t)=(1-t)(-1)+t(-i)=(t-1)-it \quad \text{where} \quad t\in[0,1] \; .$$

$$f_1(z)=\frac{(t-1)-it+1}{(t-1)-it+i}=\frac{t(1-i)}{(t-1)(1-i)}=\frac{t}{t-1} \; .$$

So $\hat{f}_1$ maps the line segment from -1 to -i to the negative real-axis. As the arcs of the circles at -1 are at angles of $\pi/4$ to the line segment and the mapping $\hat{f}_1$ is conformal then the angle between the extended lines at the origin and the negative real-axis is also $\pi/4$. Therefore the image of R under $\hat{f}_1$ is S.

(b)(iii) A conformal mapping from S onto T is the power function g(z) = z^2.

Since the combination of conformal mapping is also conformal then a conformal mapping from R to T is

$$f(z) = (g \circ f_1) = \left(\frac{z+1}{z+i}\right)^2$$

(b)(iv)

As $f_1^{-1}(w) = \dfrac{iw-1}{-w+1}$ [[D1, Sect. 2, Para. 6]]

and $g^{-1}(w) = -\sqrt{w}$, as we need the negative root,

then the inverse function $f^{-1}(z) = (f_1^{-1} \circ g^{-1})(z) = f_1^{-1}(-\sqrt{z}) = \dfrac{-i\sqrt{z}-1}{\sqrt{z}+1}$.