Question 1

(a) [[Multiplying a complex number by i = exp(iπ/2) rotates it clockwise by π/2. Drawing 1 – i and 1 + i on a Venn diagram we see that rotating 1 – i in this way gives 1 + i.]]

Since i(1- i) = 1 + i then $\left(\frac{1-i}{1+i}\right)^3 = \left(\frac{1-i}{i(1-i)}\right)^3 = \frac{1}{i^3} = i$.

[[I have included a couple of other methods.]]

$$\left(\frac{1-i}{1+i}\right)^3 = \left(\frac{\sqrt{2}\exp(-\pi/4)}{\sqrt{2}\exp(\pi/4)}\right)^3 = (\exp(-\pi/2))^3 = \exp(-3\pi/2) = i\,.$$

$$\left(\frac{1-i}{1+i}\right)^3 = \left(\frac{(1-i)^2}{(1+i)(1-i)}\right)^3 = \left(\frac{-2i}{2}\right)^3 = -i^3 = i\,.$$

(b) $\exp(2 + \pi i/6) = e^2\{\cos(\pi/6) + i\sin(\pi/6)\}$ (Unit A2, Section 4, Para. 1)

$$= \frac{e^2}{2}\left(\sqrt{3} + i\right)$$

(c) [[Parts c and d are very similar to those on the 2007 paper. In both cases the value in the bracket can be written as exp(iθ).]]

$$Log\left(\frac{1+i\sqrt{3}}{2}\right) = Log(\exp(i\pi/3)) = \frac{\pi}{3}i\,.$$

Alternatively using A2, Sect 5, Para. 1

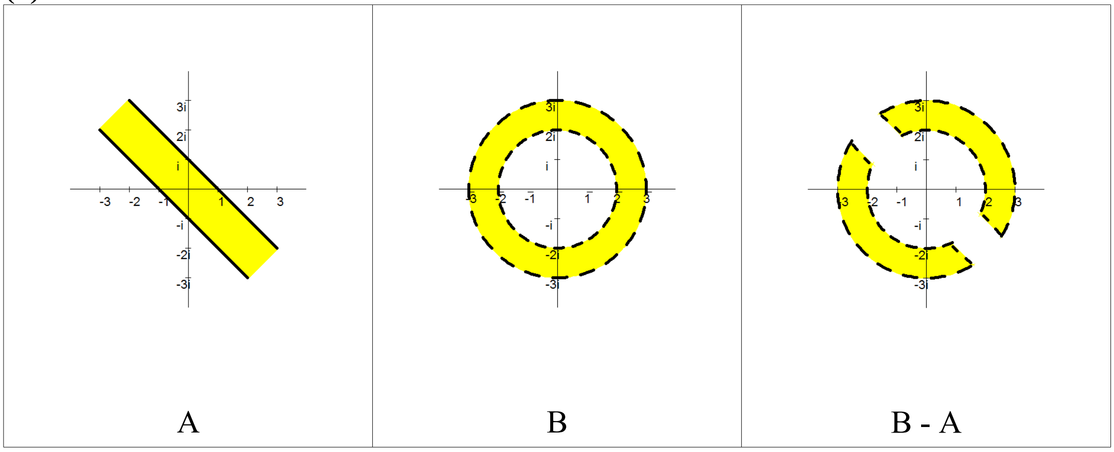
$$Log\left(\frac{1+i\sqrt{3}}{2}\right) = \log_e\left(\left|\frac{1+i\sqrt{3}}{2}\right|\right) + iArg\left(\frac{1+i\sqrt{3}}{2}\right) = \log_e 1 + \frac{\pi}{3}i = \frac{\pi}{3}i$$

(d) $\left(\frac{1+i\sqrt{3}}{2}\right)^{3-i} = \exp\left((3-i)\ Log\left(\frac{1+i\sqrt{3}}{2}\right)\right)$ (A2, Sect. 5, Para. 3)

$= \exp\left((3-i)\frac{\pi}{3}i\right) = \exp\left(i\pi + \frac{\pi}{3}\right) = -\exp\left(\frac{\pi}{3}\right)$ using result from part (c)

Question 2

(a)



(b)

	A	B	B - A
(i) open A3, Sect. 4, Para. 1	No	Yes	Yes
(ii) connected A3, Sect. 4, Para. 3	Yes	Yes	No
(iii) a region A3, Sect. 4, Paras 6-8	No	Yes	No
(iv) bounded A3, Sect. 5, Para. 4	No	Yes	Yes
(v) compact A3, Sect. 5, Para. 5	No	No	No

Question 3

(a) The standard parametrization for the line segment Γ is (A2, Sect. 2, Para. 3)

$$\gamma(t) = (1-t)\mathrm{i} + t1 = t + (1-t)i \qquad (t \in [0, 1])$$

Since γ is a smooth path and (Re z)(Im z) is continuous along the path Γ then (B1, Sect. 2, Para. 1)

$$\int_\Gamma (\operatorname{Re} z)(\operatorname{Im} z)\, dz = \int_0^1 (\operatorname{Re}\gamma(t))(Im\,\gamma(t))\,\gamma'(t)\, dt$$

$$= \int_0^1 t(1-t)(1-i)dt = (1-i)\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{1-i}{6}$$

(b) $f(z) = \dfrac{z^2-1}{\bar{z}^2+1}$ is continuous on $\mathbb{C}$ – {-i, i} and hence on the circle C.

Therefore the Estimation Theorem (Unit B1, Section 4, Para. 3) will be used to find an upper limit.

The length of C is L = 2π * 2 = 4π.

[[We hve to find an upper limit for f(z). Sometimes I get confused over whether I need to find an upper limit or lower limit for the numerator (top) and denominator of f(z). I then think of a simple example. 2/2 < 3/2 < 3/1. So we need an upper limit for the numerator (2 < 3) and a lower limit for the denominator (2 > 1).]]

Using the Triangle Inequality (Unit A1, Section 5, Para. 3b) then, for z ∈ C, we have

$$|z^2 - 1| \le |z^2| + 1 = |z|^2 + 1 = 4 + 1 = 5$$

Using the Backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 3c) then, for z ∈ C, we have

$$|\bar{z}^2 + 1| \ge \left| |\bar{z}^2| - 1 \right| = |4 - 1| = 3$$

Therefore $M = \left|\dfrac{z^2-1}{\bar{z}^2+1}\right| \le \dfrac{5}{3}$ for z ∈ C.

So $\left|\int_\Gamma \mathrm{f}(\mathrm{z})\mathrm{dz}\right| \le \mathrm{ML} = \dfrac{5}{3} * 4\pi = \dfrac{20}{3}\pi$.

Question 4

Let $\mathscr{R}$ be the simply-connected region {z: |z| < 3}.

(a)

C is a closed contour in $\mathscr{R}$, and $\dfrac{\cos z}{z-\pi}$ is analytic on $\mathscr{R}$.

By Cauchy's Theorem (B2, Sect. 1, Para. 4)

$$\int_C \frac{\cos z}{z-\pi}\,dz = 0\,.$$

(b)

As C is a simple-closed contour in $\mathscr{R}$, f(z) = cos z is analytic on $\mathscr{R}$, and $\alpha = \pi/3$ is inside C, then by Cauchy's Integral Formula (B2, Sect. 2, Para. 1)

$$\int_C \frac{\cos z}{z-\pi/3}\,dz = \int_C \frac{f(z)}{z-\pi/3}\,dz = 2\pi i f(\pi/3) = 2\pi i \cos{\pi/3} = \pi i\,.$$

(c)

As C is a simple-closed contour in $\mathscr{R}$, f(z) = cos z is analytic on $\mathscr{R}$, and $\alpha = \pi/2$ is inside C, then by Cauchy's n'th Derivative Formula (B2, Sect. 3, Para. 1) with n = 3 we have

$$\int_C \frac{\cos z}{(z-\pi/2)^4}\,dz = \frac{2\pi i}{3!} f^{(3)}(\pi/2) = \frac{\pi i}{3}\sin\frac{\pi}{2} = \frac{\pi i}{3}\,.$$

Question 5

(a)

$f(z) = \dfrac{z+1}{z(z^2+4)} = \dfrac{z+1}{z(z-2i)(z+2i)}$ has simple poles at z = 0, and z = ±2i.

$\text{Res}(f,0) = \lim_{z\to 0} (z-0)\ f(z) = \lim_{z\to 0} \dfrac{z+1}{(z^2+4)} = \dfrac{1}{4}$ [[C1, Sect. 1, Para. 1]]

$\text{Res}(f,2i) = \lim_{z\to 2i} (z-2i)\ f(z) = \lim_{z\to 2i} \dfrac{z+1}{z(z+2i)} = -\dfrac{1+2i}{8}$

$\text{Res}(f,-2i) = \lim_{z\to -2i} (z+2i)\ f(z) = \lim_{z\to -2i} \dfrac{z+1}{z(z-2i)} = -\dfrac{1-2i}{8}$

[[You may prefer the Cover-Up Rule (C1, Sect. 1, Para. 3)]]

(b)

I shall use the result given in Unit C1, Section 3, Para. 8.

Let p(t) = t + 1, q(t) = t(t^2 + 4).

p and q are polynomial functions such that the degree of q exceeds that of p by at least 2, and the pole of p/q on the real axis at z = 0 is simple. Therefore

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\,dt = \int_{-\infty}^{\infty} \frac{t+1}{t(t^2+4)}\,dt = 2\pi i S + \pi i T$$

where S is the sum of the residues of f at the poles in the upper half-plane, and
T is the sum of the residues of f at the poles on the real axis.

As S = Res(f, 2i) and T = Res(f, 0).

$$\int_{-\infty}^{\infty} \frac{t+1}{t(t^2+4)}\,dt = 2\pi i\left(-\frac{1+2i}{8}\right) + \pi i \frac{1}{4} = \frac{\pi}{2}$$

[[As it is a real integral we expect the imaginary terms to cancel]]

Question 6

(a) Let $f(z) = z^7 + 3z^5 - 1$.
The function f is analytic on the simply-connected region $\boldsymbol{R} = \mathbb{C}$ so Rouché's theorem (C2, Sect. 2, Para. 4) can be used.

Let $g_1(z) = z^7$.

Using the Triangle Inequality (A1, Sect. 5, Para. 2) when $z \in C_1 = \{ z : |z| = 2\}$ then

$$| f(z) - g_1(z) | = |3z^5 - 1 | \leq |3z^5| + | -1 | = 96 + 1 < 128 = 2^7 = | g_1(z) |.$$

Since C_1 is a simple-closed contour in $\boldsymbol{R}$ then by Rouché's theorem f has the same number of zeros as g_1 inside the contour C_1. Therefore f has 7 zeros inside C_1.

Let $g_2(z) = 3z^5$.

Using the Triangle Inequality when $z \in C_2 = \{ z : |z| = 1\}$ we have

$$| f(z) - g_2(z) | = |z^7 - 1 | \leq |z^7| + | -1 | = 1 + 1 < 3 = | g_2(z) |.$$

As C_2 is a simple-closed contour in $\boldsymbol{R}$ then by Rouché's theorem f has the same number of zeros as g_2 inside the contour C_2. Therefore f has 5 zeros inside C_2.

So f(z) has 7 - 5 = 2 solutions in the set $\{z: 1 \leq |z| < 2 \}$. Now we have to find if there are any solutions on C_2.

Since $|z_1 \pm z_2 \pm \ldots z_n| \geq |z_1| - |z_2| - \ldots -|z_n|$, (A1, Sect. 5, Para. 3(e))
then on C_2

$$|z^7 + 3z^5 - 1 | = |3z^5 + z^7 - 1| \geq |3z^5| - |z^7| - |1| = 3 - 1 - 1 > 0.$$

As f(z) is non-zero on C_2 then there are exactly 2 solutions of f(z) = 0 in the set $\{z : 1 < |z| < 2 \}$.

(b) $f'(z) = 7z^6 + 15z^4$. On the real-axis $f'(z) > 0$ when $z \neq 0$ and so f(z) is increasing except at z = 0. Since f(0) = -1 and f(1) = 3 then f has only one real solution. Therefore the other 6 solutions are imaginary.

If α is a solution of $z^7 + 3z^5 - 1 = 0$ then taking the complex conjugate of both sides of the equation gives $\overline{\alpha}^7 + 3\overline{\alpha}^5 - 1 = 0$. So both α and its conjugate are solutions of the equation

As one member of each of the 3 conjugate pairs lies above the real-axis then there are 3 solutions in the upper half-plane.

Question 7

(a)

The conjugate velocity function $\overline{q}(z) = -i/z^2$.
As q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C} - \{0\}$ and $\overline{q}$ is analytic on $\mathbb{C} - \{0\}$ then q is a model fluid flow (Unit D2, Section 1, Para. 14).

(b) On $\mathbb{C} - \{0\}$, $\Omega(z) = \frac{i}{z}$ is a primitive of $\overline{q}$. Therefore Ω is a complex potential function for the flow (Unit D2, Section 2, Para. 1).

The stream function $\Psi(x, y) = \operatorname{Im}\Omega(z)$ (Unit D2, Section 2, Para. 4)

$$= \operatorname{Im}\left(\frac{i}{x+iy}\right) = \operatorname{Im}\left(\frac{ix+y}{x^2+y^2}\right) = \frac{x}{x^2+y^2}$$, where z = x + iy, (x,y) ≠ (0, 0)

A streamline through the point 2 satisfies the equation

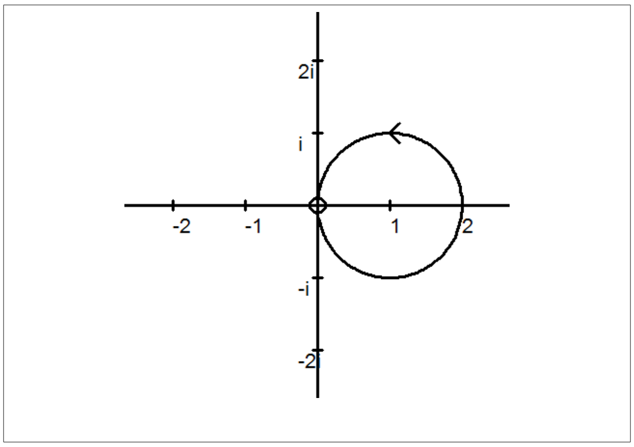
$$\frac{x}{x^2+y^2} = \Psi((2,0)) = \tfrac{1}{2}$$ (Unit D2, Section 2, Para. 4)

Therefore the streamline through i has the equation $x^2 - 2x + y^2 = 0$ or

$$(x-1)^2 + y^2 = 1$$

Since q((2,0)) = i/4 (positive y direction) then the direction of flow is as shown.

[[As q is not defined at 0 the origin is omitted from the circle in the diagram below.]]



(c) If C_Γ is the circulation of along Γ and F_Γ is the flux of q across Γ then (D1, Sect. 1, Para. 1 and D2, Sect. 1, Paras. 9 & 10)

$$C_\Gamma + iF_\Gamma = \int_\Gamma \overline{q}(z)dz = \Omega((4,0)) - \Omega((2,0)) = \frac{i}{4} - \frac{i}{2} = -\frac{i}{4}$$

Therefore the flux of q across Γ is -¼.

[[The normal is in the direction -i.]]

Question 8

(a)

Using the result in Unit D3, Section 2, Para. 1 then the iteration sequence $z_{n+1} = 2z_n^2 - 4z_n + 2$ is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + (2*2 + (-4)/2 - (-4)^2/4) = w_n^2 - 2$$

and conjugating function $h(z) = 2z - 2$.

Therefore $w_0 = h(z_0) = 2z_0 - 2 = 2 - 2 = 0$. (Unit D3, Section 1, Para. 7).

(b)

If □ is a fixed point of P_{-2} then $P_{-2}(□) = □^2 - 2 = □$ (D3, Sect. 1, Para 3).
As $□^2 - □ - 2 = (□ + 1)(□ - 2) = 0$ then $P_{-2}(z)$ has fixed points at $z = -1$ and $z = 2$.

$P_{-2}'(z) = 2z$.

As $|P_{-2}'(-1)| = 2 > 1$ and $|P_{-2}'(2)| = 4 > 1$ then both are repelling fixed points (D3, Sect. 1, Para. 5).

(c) [[If you have add coordinates on the axes of the diagram of the Mandelbrot set then you will see that c is not in the Mandelbrot set.]]

The Mandelbrot set M can be specified as

$$\{c : |P_c^n(0)| \le 2, \text{for } n = 1, 2, \ldots\}$$

where $P_c(z) = z^2 + c$. (D3, Sect. 4, Para. 5 and D3, Sect. 2, Para. 2).

Let $c = \frac{1}{2} + i$

$P_c(0) = c = \frac{1}{2} + i$.
$P_c^2(0) = \left(\frac{1}{2} + i\right)^2 + \left(\frac{1}{2} + i\right) = \frac{1}{4} - 1 + i + \frac{1}{2} + i = -\frac{1}{4} + 2i.$

As $|P_c^2(0)| > 2$ then c does not lie in the Mandelbrot set.

Question 9

(a)(i)

$$f(x+iy) = 2e^{i\operatorname{Re} z} - \bar{z} = 2e^{ix} - (x - iy) = (2\cos x - x) + i(2\sin x + y) = u(x,y) + iv(x,y)$$

where u(x,y) = 2 cos x - x, and v(x,y) = 2 sin x + y are real-valued functions .

(a)(ii)

f is defined on the region $\mathbb{C}$.

$$\frac{\partial u}{\partial x} = -2\sin x - 1, \qquad \frac{\partial u}{\partial y} = 0, \qquad \frac{\partial v}{\partial x} = 2\cos x, \qquad \frac{\partial v}{\partial y} = 1$$

If f is differentiable at (a, b) ∈ $\mathbb{C}$, then the Cauchy-Riemann equations hold (A4, Sect. 2, Para. 1) so

$$\frac{\partial u}{\partial x}(a,b) = -2\sin a - 1 = 1 = \frac{\partial v}{\partial y}(a,b), \text{ and}$$

$$\frac{\partial v}{\partial x}(a,b) = 2\cos a = 0 = -\frac{\partial u}{\partial y}(a,b)$$

So sin a = -1 and cos a = 0. **(A)**

Therefore the Cauchy-Riemann equations hold on the set

S = {(a, b) : a = (2n + 3/2)π for any integer n, any real number b}.

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous on S.
3. satisfy the Cauchy-Riemann equations on S

then, by the Cauchy-Riemann Converse Theorem (A4, Sect. 2, Para. 3), f is differentiable on S.

(a)(iii) If (a, b) ∈ S, then using equations (A)

$$f'(a, b) = \frac{\partial u}{\partial x}(a,b) + i\frac{\partial v}{\partial x}(a,b) = (-2\sin a - 1) + i2\cos a = 1$$ (A4, Sect. 2, Para. 3).

Therefore f' is constant on S.

(b)(i) The domain of g is $\mathbb{C}$ (Unit A4, Section 1, Para. 7) and its derivative $g'(z) = 2z$ also has domain $\mathbb{C}$ (Unit A4, Section 3, Para. 4) . Therefore g is analytic on $\mathbb{C} - \{0\}$. Since $g'(z) \neq 0$ on $\mathbb{C} - \{0\}$ then g is conformal on $\mathbb{C} - \{0\}$ (Unit A4, Section 4, Para. 6).

(b)(ii) As g is analytic on $\mathbb{C}$ and $g'(2i) \neq 0$ then a small disc centred at 2i is mapped approximately (Unit A4, Section 1, Para. 11) to a small disc centred at $g(2i) = -4 + 2 = -2$. The disc is rotated by $\text{Arg}\,(g'(2i)) = \text{Arg}\,4i = \pi/2$, and scaled by a factor $|g'(2i)| = |4i| = 4$.

(b)(iii) 1 is in the domain of γ_1 and $\gamma_1(1) = 1 - 1 + 2i = 2i$.

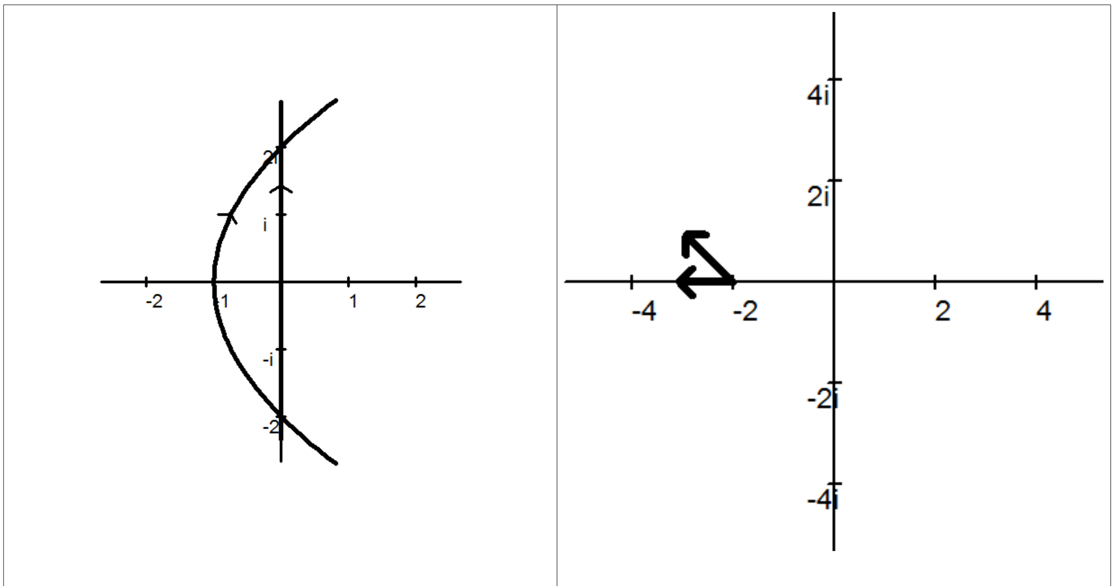
0 is in the domain of γ_2 and $\gamma_2(0) = (0 + 2)i = 2i$. Therefore Γ_1 and Γ_2 meet at the point 2i.

As $\gamma_1'(t) = 2t + 2i$ then at 1, $Arg(\gamma_1'(1)) = Arg(2(1+i)) = \frac{\pi}{4}$.

As $\gamma_2'(t) = 3i$ then at 0, $Arg(\gamma_2'(0)) = Arg(3i) = \frac{\pi}{2}$.

Therefore the angle from Γ_1 to Γ_2 at their point of intersection is $\pi/4$.

(b)(iv) $\gamma_1(t) = x + iy$, where $x = t^2 - 1$ and $y = 2t$. Therefore $y^2 = 4t^2 = 4(x + 1)$. So Γ_1 is a parabola.



(b)(iv) (b)(v)

In diagram (b)(iv) the imaginary-axis is Γ_2.
In diagram (b)(v) the horizontal line is $g(\Gamma_2)$.

Question 10

(a)

The domain of f, A = $\mathbb{C} - \{2n\pi : n \text{ is an integer}\}$. [[as $\cos(2n\pi) = 1$]]

(b)

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \ldots$, for $z \in \mathbb{C}$. (B3, Sect. 3, Para. 5)

So $1 - \cos z = \frac{z^2}{2}\left(1 - \frac{2z^2}{4!} + \frac{2z^4}{6!} \ldots\right) = \frac{z^2}{2}\left(1 - \frac{z^2}{12} + \frac{z^4}{360} \ldots\right)$.

So
$$\frac{z}{1-\cos z} = \frac{2}{z}\left(1 - \frac{z^2}{12} + \frac{z^4}{360} - \ldots\right)^{-1}$$
$$= \frac{2}{z}\left\{1 + \left(\frac{z^2}{12} - \frac{z^4}{360} + \ldots\right) + \left(\frac{z^2}{12} - \frac{z^4}{360} + \ldots\right)^2 + \ldots\right\}$$
$$= \frac{2}{z}\left\{1 + \frac{z^2}{12} + z^4\left(-\frac{1}{360} + \frac{1}{144}\right) + \ldots\right\}$$

As $-\frac{1}{360} + \frac{1}{144} = \frac{1}{720}(-2+5) = \frac{1}{240}$ then the Laurent series about 0 for f is

$$\frac{2}{z} + \frac{z}{6} + \frac{z^3}{120} + \ldots = \sum_{n=-\infty}^{\infty} a_n z^n \text{ for } 0 < |z| < 2\pi$$

As C is a circle with centre 0 then (B4, Sect. 4, Para. 2)

$$\int_C f(z)\,dz = 2\pi i a_{-1} = 2\pi i(2) = 4\pi i.$$

(c) [[Is this correct?]]

Suppose that g is another analytic function with domain A which agrees with f on $\{iy : y > 0\}$

The set $S = \left\{ i\left(1+\frac{1}{n}\right) : n = 1,2,3,\ldots \right\} \subseteq A$ and has the limit point $i \in A$.

f agrees with g throughout the set $S \subseteq A$ and S has a limit point which is in A. By the Uniqueness theorem (B3, Sect. 5, Para. 7) f agrees with g throughout A. Hence f is the only analytic function with domain A such that $f(iy) = \dfrac{iy}{1-\cosh y}$ for $y > 0$.

(d)

Since $\cos z = 1$ when $z = 0$, $z = \pm 2\pi$, $z = \pm 4\pi$, …. then f(z) has singularities at points of the form $2k\pi$, $k \in \mathbb{Z}$.

Singularity at z = 0 (k = 0).

At z = 0 we can use the Laurent series found in part (a). Since $\lim_{z\to 0} zf(z) = 2$ then the singularity at 0 is a pole of order 1 (B4, Sect. 3, Para. 2).

Singularities at z = 2kπ where k ∈ $\mathbb{Z}$ – {0}.

$$f(z) = \frac{z}{1-\cos z} = \frac{z-2k\pi}{1-\cos(z-2k\pi)} + \frac{2k\pi}{1-\cos(z-2k\pi)}$$

Since

$$\lim_{z\to 2k\pi} (z-2k\pi)^2 f(z) = \lim_{z\to 2k\pi}\left\{ \frac{(z-2k\pi)^2}{1-\cos(z-2k\pi)} + \frac{2k\pi\,(z-2k\pi)^2}{1-\cos(z-2k\pi)} \right\} = 0 + 4k\pi$$

then there is a pole of order 2 at z = 2kπ (B4, Sect. 3, Para. 2).

Question 11

(a)

Let $D_f = \{z: |z| < 3\}$ and $D_g = \{z: |z| > 3\}$. D_f and D_g are regions.
Since $D_f \cap D_g = \varnothing$ then f and g are not direct analytic continuations of each other (C3, Sect. 1, Para. 1).

When $z \in D_f$ then $|z|/3 < 1$ and the geometric series $\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$ is convergent and has the sum

$$\frac{1}{1-\frac{z}{3}} = \frac{3}{3-z}.\quad \text{(B3, Sect. 3, Para. 5)}$$

When $z \in D_g$ then $3/|z| < 1$ and the geometric series $\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n$ is convergent and has the sum $\frac{1}{1-\frac{3}{z}} = \frac{z}{z-3}$. So $-\sum_{n=1}^{\infty}\left(\frac{3}{z}\right)^n = -\frac{3}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n = -\frac{3}{z}\left(\frac{z}{z-3}\right) = \frac{3}{3-z}$.

Let $h(z) = \frac{3}{3-z}$ on D_h, where $D_h = \mathbf{C} - \{3\}$.

Since f = h when $z \in D_f \subseteq D_f \cap D_h$ then h is a direct analytic continuation of f.

Since g = h when $z \in D_g \subseteq D_g \cap D_h$ then g is a direct analytic continuation of h.

Since (f, D_f), (g, D_g), (h, D_h) form a chain then f and g are indirect analytic continuations of each other (C3, Sect. 2, Para. 3).

(b)

Let $f(z) = z^2 \exp(1 + z^2)$ and $R = \{z : |z| < 2\}$.

[[The boundary ∂R is defined in A3, Sect. 5, Para. 10.]]

As f is analytic on the bounded region R and continuous on its closure $\overline{R}$ (C2, Sect. 4, Para. 3) then, by the Maximum Principle (C2, Sect. 4, Para. 4), there exists an

$\alpha \in \partial R = \{ z : |z| = 2\}$ such that

$|f(z)| \le |f(\alpha)|$ for $z \in \overline{R}$.

When $z \in \partial R$ we can write it in the form $z = 2\exp(i\theta)$, where θ is real.

Then $|z^2 \exp(1 + z^2)| = |z^2|\ |\exp(1 + z^2)|$ (A1, Sect. 1, Para. 8)

$= 4\exp(\mathrm{Re}(1 + z^2))$ (A2, Sect. 4, Para. 2)

$= 4\exp(\mathrm{Re}(1 + 4\exp(i2\theta)))$

$= 4\exp(1 + 4\cos 2\theta)$.

This is a maximum when $\cos 2\theta = 1$. So 2θ is a multiple of 2π.

Therefore $\max\{ |z^2 \exp(1 + z^2)| : |z| \le 2 \} = 4e^5$.
The maximum is attained when

$z = 2e^0 = 2$, and $z = 2e^{i\pi} = -2$.

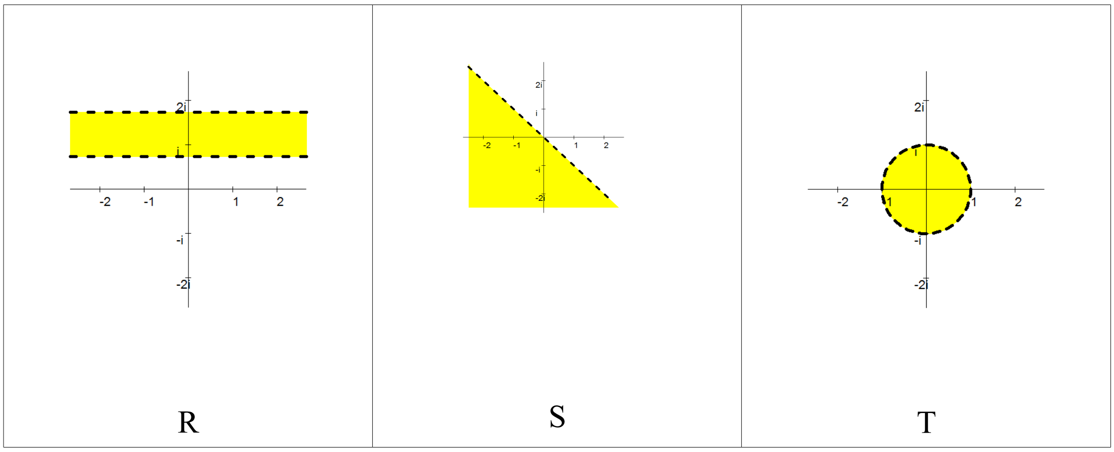
Question 12

(a)

Using the formula for a transformation mapping points to the standard triple (D1, Sect. 2, Para. 11) then the Möbius transformation $\hat{f}_1$ which maps -1 - i, 0 , and 1 + i to the standard triple of points 0, 1, and ∞ respectively is

$$f_1(z) = \frac{(z-(-1-i))}{(z-(1+i))}\frac{(0-(1+i))}{(0-(-1-i))} = \frac{-z-(1+i)}{z-(1+i)}$$

(b)(i)



(b)(ii) Let C be the boundary of S. Then C is a generalised circle (D1, Sect. 1, Para. 10), and -1 – i and 1 + i are inverse points of C since, when z ∈ C,
| z - (-1 – i) | = | z - (1 + i) | (D1, Sect. 3, Para. 4)

Therefore $\hat{f}_1(-1-i) = 0$ and $\hat{f}_1(1+i) = \infty$ are inverse points with respect to $\hat{f}_1(C)$ (D1, Sect. 3, Para. 6). So $\hat{f}_1(C)$ is a circle with centre 0 (D1, Sect. 3, Para. 5) and, as $\hat{f}_1(0) = 1$, of radius 1. Therefore $\hat{f}_1(S) = T$.

[[Before I consulted the handbook I said a general point on the boundary of S was a – ia where a is real.
As $\hat{f}_1(a-ia) = \frac{-a+ia-1-i}{a-ia-1-i} = \frac{-(a+1)+i(a-1)}{(a-1)-i(a+1)}$ then a – ia is mapped to a point on the unit circle as $\left|\hat{f}_1(a-ia)\right| = 1$. As extended Mobius transformations map generalised circles onto generalized circles that the boundary of S is mapped onto the unit circle. As the point -1 – i ∈ S and is mapped to 0 then $0 \in \hat{f}_1(S)$. Therefore $\hat{f}_1(S) = T$.
]]

(b)(iii) [[Unit D1, Sect. 4, Para. 5 shows the effect of exp z on $\{z : -\pi/2 < \operatorname{Im} z < \pi/2\}$.]]

If w ∈ R then

$$\exp(w) = \exp(z + 5\pi i/4) = \exp(5\pi i/4)\exp(z), \text{ where } z \in \{z : -\pi/2 < \operatorname{Im} z < \pi/2\}.$$

Therefore the image of R may be found by finding the image of $\{z : -\pi/2 < \operatorname{Im} z < \pi/2\}$ and then rotating it counter-clockwise about the origin by 5π/4. Using the figure in D1, Sect. 4, Para. 5 it is apparent that exp(R) = S.

[[I find it easier to imagine a clockwise rotation by 3π/4.]]

So a conformal mapping from f(R) onto S is g(z) = exp(z).

Since the combination of conformal mapping is also conformal then a conformal mapping from R to T is

$$f(z) = (f_1 \circ g)(z) = f_1(\exp(z)) = \frac{-e^z - (1+i)}{e^z - (1+i)}.$$

(b)(iv) As $(g^{-1} \circ g)(z) = z$ then $g^{-1}(z) = \operatorname{Log} z + 2\pi i$.

Since $f^{-1} = (f_1 \circ g)^{-1} = (g^{-1} \circ f_1^{-1})$ then using Unit D1, Section 2, Para. 6 we have

$$f^{-1}(w) = Log\left(f_1^{-1}(w)\right) + 2\pi i = Log\frac{-(1+i)w + (1+i)}{-w-1} + 2\pi i = Log\left((1+i)\frac{w-1}{w+1}\right) + 2\pi i$$

(b)(v) Therefore

$$p = f^{-1}(0) = Log(-(1+i)) + 2\pi i = \log_e|-1-i| + iArg(-1-i) + 2\pi i = \log_e\sqrt{2} + i\tfrac{5}{4}\pi .$$

Not every conformal mapping from R to T maps p to 0.

If a is real then, for any point w ∈ R, the point w + a ∈ R.
So the function $g_1(z) = \exp(z + a)$ also maps R to S and $(f_1 \circ g_1)(z)$ maps R to T.
Therefore if a is non-zero then the conformal mapping $(f_1 \circ g_1)$ will map another point, p – a, to 0.