Question 1

(a) $\frac{7-4i}{2+i} = \frac{7-4i}{2+i}\,\frac{2-i}{2-i} = \frac{14-4-8i-7i}{2^2+1^2} = \frac{10-15\mathrm{i}}{5} = 2-3i$. (A1, Sec. 1, Par. 6)

(b) $2\mathrm{e}^{-\mathrm{i}\pi/6} = 2\left(\cos\left(-\tfrac{\pi}{6}\right) + i\sin\left(-\tfrac{\pi}{6}\right)\right) = 2\left(\cos\tfrac{\pi}{6} - i\sin\tfrac{\pi}{6}\right)$ (A2, Sect. 4, Para. 1)

$= 2\left(\tfrac{\sqrt{3}}{2} - i\tfrac{1}{2}\right) = \sqrt{3} - i$.

(c) Using the result from part (b) we have

$\left(\sqrt{3}-i\right)^6 = \left(2e^{-i\pi/6}\right)^6 = 2^6 e^{-i\pi} = -64$.

(d) $i^{-2i} = \exp\left(-2iLog(i)\right)$. (A2, Sect. 5, Para. 3)

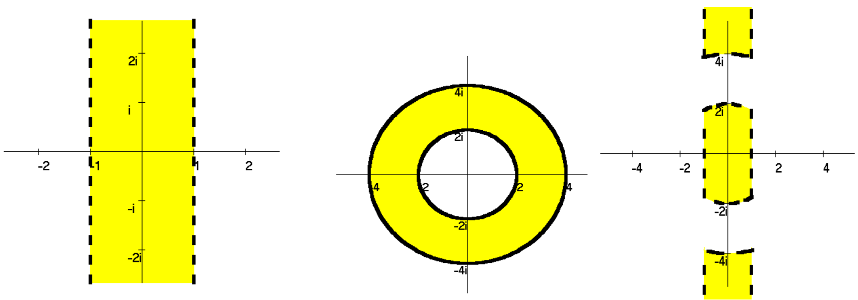
$\mathrm{Log}(\mathrm{i}) = \log_e|i| + iArg(i)$ (A2,Sect. 5, Para. 1)

$= \log_e 1 + i\pi/2 = i\pi/2$.

So $i^{-2i} = \exp\left(-2i * i\pi/2\right) = e^{\pi}$.

Question 2

(a)



A B C = A - B

(b)(i) A is a region. (A3, Sect. 4, Para.6)
B is not a region as it is not open.
C is not a region as it is not connected.

(b)(ii)
A and C are not compact (A3, Sect. 5, Para. 5) as they are neither closed nor bounded.
B is compact.

Question 3

(a)

(a)(i) The standard parametrization for the line segment Γ is (A2, Sect. 2, Para. 3)

$$\gamma(t) = (1-t)(-1) + ti = (t-1) + it \qquad (t \in [0, 1])$$

(a)(ii) Since γ is a smooth path and $(\text{Im } z)^2$ is continuous along the path Γ then (B1, Sect. 2, Para. 1)

$$\int_\Gamma (\text{Im}\, z)^2\, dz = \int_0^1 (\text{Im}\,\gamma(t))^2 \gamma'(t)\, dt = \int_0^1 t^2(1+i)dt = (1+i)\left[\frac{t^3}{3}\right]_0^1 = \frac{1+i}{3}$$

(b)

As $f(z) = \dfrac{3\exp(\bar{z})}{3+z^5}$ is continuous on the line Γ then we can use the Estimation Theorem (B1, Sect. 4, Para. 3) to obtain an upper estimate for the modulus of the integral.

The length of Γ is $L = |i - (-1)| = \sqrt{2}$.

$|3\exp(\bar{z})| = 3\exp(\text{Re}\,\bar{z}) = 3\exp(\text{Re}\, z)$. (A2, Sect. 4, Para. 2)

On Γ, $\text{Re } z \le 0$ and hence $|3\exp(\bar{z})| \le 3e^0 = 3$.

Using the Backwards form of the Triangle Inequality (A1, Sect. 5, Para. 3c) then

$$|3+z^5| \ge \big|\,|3| - |-z^5|\,\big| = \big|\,3 - |z|^5\,\big|.$$

Since no part of Γ lies outside the closed disc $\{z : |z| \le 1\}$ then on Γ we have $|z| \le 1$. Therefore

$$|3+z^5| \ge |3 - 1^5| = 2.$$

Therefore $M = \left|\dfrac{3e^{\bar{z}}}{3+z^5}\right| \le \dfrac{3}{2}$ for $z \in \Gamma$.

Therefore by the Estimation Theorem an upper estimate for the modulus of the integral is

$$ML = \frac{3}{2}\sqrt{2}.$$

Question 4

Let **R** = {z: |z| < 1}.

(a)

R is a simply-connected region, $\dfrac{\cos 2z + \sin 2z}{(z-i)^2}$ is analytic on **R,** and C is a closed contour in **R**. So by Cauchy's Theorem (B2, Sect. 1, Para. 4)

$$\int_C \frac{\cos 2z + \sin 2z}{(z-i)^2}\,dz = 0.$$

(b)

R is a simply-connected region, C is a simple-closed contour in **R,**
$f(z) = \dfrac{\cos 2z + \sin 2z}{z-i}$ is analytic on **R,** and 0 is inside C. So using Cauchy's Integral Formula (B2, Sect. 2, Para. 1) we have

$$\int_C \frac{\cos 2z + \sin 2z}{z(z-i)}\,dz = \int_C \frac{f(z)}{z-0}\,dz = 2\pi i f(0) = 2\pi i\,\frac{\cos 0 + \sin 0}{0-i} = -2\pi\ .$$

(c)

R is a simply-connected region, C is a simple-closed contour in **R,**
$f(z) = \cos 2z + \sin 2z$ is analytic on **R** and and 0 is inside C. So using Cauchy's n'th Derivative Formula (B2, Sect. 3, Para. 1) we have

$$\int_C \frac{\cos 2z + \sin 2z}{z^2}\,dz = \int_C \frac{f(z)}{(z-0)^2}\,dz$$
$$= \frac{2\pi i}{1!} f^{(1)}(0) = 2\pi i\{-2\sin 0 + 2\cos 0\} = 4\pi i.$$

Question 5

(a)

f has simple poles at z = 0, z = 3 and z = 1/3.

$$\text{Res}(f,0) = \lim_{z \to 0} (z-0)\, f(z) = \lim_{z \to 0} \frac{z^2+1}{(z-3)(3z-1)}$$ [[C1, Sect. 1, Para. 1]]

$$= \frac{0+1}{(0-3)(0-1)} = \frac{1}{3}.$$

$$\text{Res}(f,3) = \lim_{z \to 3} (z-3)\, f(z) = \lim_{z \to 3} \frac{z^2+1}{z(3z-1)}$$

$$= \frac{3^2+1}{3(9-1)} = \frac{5}{12}.$$

$$\text{Res}(f,1/3) = \lim_{z \to 1/3} (z-1/3)\, f(z) = \lim_{z \to 1/3} \frac{z^2+1}{z(z-3)3}$$

$$= \frac{\frac{1}{3}^2+1}{\frac{1}{3}\left(\frac{1}{3}-3\right)3} = \frac{1+9}{3-27} = -\frac{5}{12}.$$

(b)

I shall use the strategy given in C1, Sect. 2, Para. 2.

$$\int_0^{2\pi} \frac{\cos t}{5-3\cos t}\,dt = \int_C \frac{\frac{1}{2}\left(z+z^{-1}\right)}{5-3\frac{1}{2}\left(z+z^{-1}\right)} \frac{1}{iz}\,dz$$, where C is the unit circle {z : |z| = 1}.

$$= -i\int_C \frac{z^2+1}{\left(10z-3z^2-3\right)z}\,dz = i\int_C \frac{z^2+1}{z(z-3)(3z-1)}\,dz$$

The singularities of f(z) inside the unit circle C are at z = 0 and z = 1/3.

Therefore

$$\int_0^{2\pi} \frac{\cos t}{5-3\cos t}\,dt = i*2\pi i\left\{\text{Res}(\text{f},0)+\text{Res}\left(\text{f},\tfrac{1}{3}\right)\right\} = -2\pi\left(\tfrac{1}{3}-\tfrac{5}{12}\right) = \frac{\pi}{6}.$$

Question 6

(a)

The function f is analytic on the simply-connected region $\boldsymbol{R} = \mathbb{C}$ so Rouché's theorem (C2, Sect. 2, Para. 4) can be used.

(a)(i) Let $g_1(z) = z^7$.

Using the Triangle Inequality (A1, Sect. 5, Para. 3) when $z \in C_1$ then

$$| f(z) - g_1(z) | = |4z^3 - 2i| \leq |4z^3| + |-2i| = 32 + 2 < 2^7 = | g_1(z) |.$$

Since C_1 is a simple-closed contour in $\boldsymbol{R}$ then by Rouché's theorem f has the same number of zeros as g_1 inside the contour C_1. Therefore f has 7 zeros inside C_1.

(a)(ii) Let $g_2(z) = 4z^3$.

Using the Triangle Inequality when $z \in C_2$ we have

$$| f(z) - g_2(z) | = |z^7 - 2i| \leq |z^7| + |-2i| = 1 + 2 < 4 = | g_2(z) |.$$

As C_2 is a simple-closed contour in $\boldsymbol{R}$ then by Rouché's theorem f has the same number of zeros as g_2 inside the contour C_2. Therefore f has 3 zeros inside C_2.

(b)

From part(a) f(z) has 7 - 3 = 4 solutions in the set $\{z: 1 \leq |z| < 2\}$. Therefore we have to find if there are any solutions on C_2.

Since $|z_1 \pm z_2 \pm \ldots z_n| \geq |z_1| - |z_2| - \ldots - |z_n|$, (A1, Sect. 5, Para. 3(e))
then on C_2

$$|z^7 + 4z^3 - 2i| = |4z^3 + z^7 - 2i| \geq |4z^3| - |z^7| - |2i| = 4 - 1 - 2 > 0.$$

As f(z) is non-zero on C_2 then there are exactly 4 solutions of f(z) = 0 in the set $\{z: 1 < |z| < 2\}$.

Question 7

(a)

q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C}$ and the conjugate velocity function $\overline{q}(z) = z + i$ is analytic on $\mathbb{C}$. Therefore q is a model flow on $\mathbb{C}$ (Unit D2, Section 1, Para. 14).

(b)

The complex potential function Ω is given by

$$\Omega'(z) = \overline{q}(z) = z + i. \qquad \text{(D2, Sect. 2, Para. 1)}$$

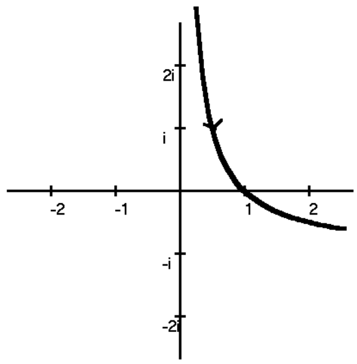
Therefore the complex potential function .

$$\Omega(z) = \frac{z^2}{2} + iz$$

The stream function (Unit D2, Section 2, Para. 4)

$$\Psi(x,y) = \operatorname{Im}\Omega(z) = \operatorname{Im}\left(\frac{x^2 - y^2 + i2xy}{2} + i(x + iy)\right) = xy + x$$

A streamline through 1 is given by $x(y+1) = \Psi(1,0) = 1$.
So x and y are related by the equation $y = (1/x) - 1$.
The direction of flow at 1 is given by the angle Arg q(1) = Arg 1 – i = - π/4



(c)

As q is a model flow velocity function on the region $\mathbb{C}$ and Γ lies in $\mathbb{C}$ then the circulation of q along Γ is, using the result given in D2, Sect. 2, Para. 1,

Re ($\Omega(\beta) - \Omega(\alpha)$), where α and β are the start and end points of Γ.

As $\alpha = 0$ and $\beta = 2$ then the required circulation is

$$\operatorname{Re}\left(\frac{2^2}{2} + i2 - 0\right) = 2.$$

Question 8

(a)

If α is a fixed point of f then $f(\alpha) = \alpha^2 + 4\alpha + 2 = \alpha$ (D3, Sect. 1, Para 3).
As $\alpha^2 + 3\alpha + 2 = (\alpha + 1)(\alpha + 2) = 0$ then f(z) has fixed points at z = -1 and z = -2.

$f'(z) = 2z + 4$.

As $|f'(-1)| = 2 > 1$ then -1 is a repelling fixed point (D3, Sect. 1, Para. 5).
As $f'(-2) = 0$ then -2 is a super-attracting fixed point.

(b)(i)

$c = -\frac{3}{2} + \frac{1}{2}i$

[[If you have added coordinates on the axes of the diagram of the Mandelbrot set then you will see that c is not in the Mandelbrot set.]]

$P_c(0) = -\frac{3}{2} + \frac{1}{2}i$.
$P_c^2(0) = \left(-\frac{3}{2} + \frac{1}{2}i\right)^2 + \left(-\frac{3}{2} + \frac{1}{2}i\right) = \frac{9}{4} - \frac{1}{4} - \frac{3}{2}i - \frac{3}{2} + \frac{1}{2}i = \frac{1}{2} - i.$
$P_c^3(0) = \left(\frac{1}{2} - i\right)^2 + \left(-\frac{3}{2} + \frac{1}{2}i\right) = \frac{1}{4} - 1 - i - \frac{3}{2} + \frac{1}{2}i = -\frac{9}{4} - \frac{1}{2}i.$
As $\left|P_c^3(0)\right| > 2$ then c does not lie in the Mandelbrot set (D3, Sect. 4, Para. 5).

(b)(ii)

$c = -1 - \frac{1}{5}i$

$\left|c + 1\right| = \left|-\frac{1}{5}i\right| < \frac{1}{4}$. Therefore P_c has an attracting 2-cycle (D3, Sect. 4, Para. 9(b)).
Hence c belongs to the Mandelbrot set (D3, Sect. 4, Para. 8).

Question 9

(a)

(a)(i)

$f(z) = 2z + |z|^2 = f(x+iy) = 2(x+iy) + (x^2 + y^2) = u(x,y) + iv(x,y)$,
where $u(x,y) = 2x + x^2 + y^2$, and $v(x,y) = 2y$.

(a)(ii)

f is defined on the region $\mathbb{C}$.

$\frac{\partial u}{\partial x} = 2 + 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 2$

If f is differentiable then the Cauchy-Riemann equations hold (A4, Sect. 2, Para. 1). If they hold at (a, b) then

$\frac{\partial u}{\partial x}(a,b) = 2 + 2a = 2 = \frac{\partial v}{\partial y}(a,b)$, and

$\frac{\partial v}{\partial x}(a,b) = 0 = -2b = -\frac{\partial u}{\partial y}(a,b)$

Therefore the Cauchy-Riemann equations only hold at (0, 0).

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous at (0, 0).
3. satisfy the Cauchy-Riemann equations at (0, 0)

then, by the Cauchy-Riemann Converse Theorem (A4, Sect. 2, Para. 3), f is differentiable at 0.

As the Cauchy-Riemann equations only hold at (0, 0) then f is not differentiable on any region surrounding 0. Therefore f is not analytic at 0. (A4, Sect. 1, Para. 3)

(a)(iii)

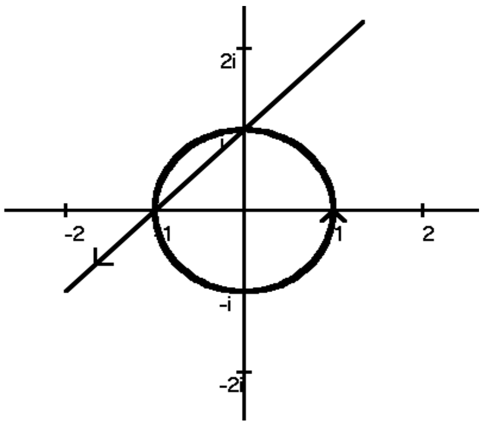
$f'(0, 0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 2$ (A4, Sect. 2, Para. 3).

(b)(i)

g(z) is analytic on the region $\mathbb{C} - \{0\}$ (Unit A4, Section 3, Para. 4),
and $g'(z) = -\dfrac{3i}{z^4}$ on $\mathbb{C} - \{0\}$.

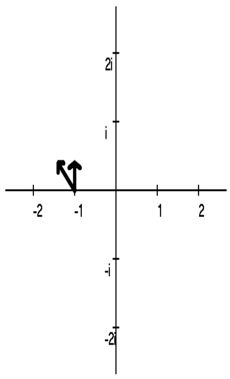
As $g'(i) = -\dfrac{3i}{i^4} = -3i \neq 0$ and g is analytic at i , then g is conformal at z = i.
(Unit A4, Section 4, Para. 6)

(b)(ii) $\pi/2$ is in the domain of γ_1 and $\gamma_1(\pi/2) = e^{i\pi/2} = i$.
0 is in the domain of γ_2 and $\gamma_2(0) = i$. Therefore Γ_1 and Γ_2 meet at the point i.



(b)(iii)
As g is analytic on $\mathbb{C}$- {0} and g' (i) ≠ 0 then a small disc centred at i is mapped approximately (Unit A4, Section 1, Para. 11) to a small disc centred at g(i) = -1. The disc is rotated by Arg (g '(i)) = Arg -3i = $-\pi/2$, and scaled by a factor | g '(i)| = 3.

In the diagram below $g(\Gamma_1)$ is the vertical line.



Question 10

(a)

$$f(z) = \frac{\sin z}{z(z-3)^4}.$$

The singularities occur when the denominator of f is zero. Therefore there are singularities at z = 0 and z = 3.

As $\lim_{z \to 0}(z-0)f(z) = \frac{\sin 0}{(-3)^4} = 0$ then the singularity at 0 is removable (Unit B4, Section 3, Para. 1(D)).

As $\lim_{z \to 3}(z-3)^4 f(z) = \frac{\sin 3}{3} \neq 0$ then f has a pole of order 4 at 3. (Unit B4, Section 3, Para. 2(B)).

(b)(i) The Laurent Series for g(z) = sin (1/z) about 0 is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \frac{1}{7!z^7} + \dots$$

The annulus of convergence is $\{ z : 0 < |z| < \infty \}$.

(b)(ii)

$$z^4 \sin(1/z) = z^3 - \frac{z}{3!} + \frac{1}{5!z} - \frac{1}{7!z^3} + \dots = \sum_{n=-\infty}^{\infty} a_n z^n.$$

$z^4 \sin(1/z)$ is analytic on the punctured disc $\mathbb{C} - \{0\}$.

As C is a circle with centre 0 then (Unit B4, Section 4, Para. 2)

$$\int_C z^4 \sin\left(\frac{1}{z}\right) dz = 2\pi i \, a_{-1} = 2\pi i \left(\frac{1}{5!}\right) = \frac{\pi i}{60}.$$

(c)(i)

The Taylor series (Unit B3, Section 3, Para. 5) around 0 for cosh z and Log(1 + z) are

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots \qquad \text{for } z \in \mathbb{C},$$

and $$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots \qquad \text{for } |z| < 1.$$

h(z) = Log (cos z) = Log (1 + [cos z - 1]).

When z = 0 then (cos z - 1) = 0. Therefore we can expand the series about 0 using the Composition Rule (Unit B3, Section 4, Para. 3) when |cos z - 1| < 1.

$$\begin{aligned}
h(z) &= \left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots\right) - \frac{1}{2}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right)^2 + \frac{1}{3}\left(-\frac{z^2}{2!} + \ldots\right)^3 + \ldots \\
&= -\frac{z^2}{2} + z^4\left(\frac{1}{4!} - \frac{1}{2}\frac{1}{(2!)^2}\right) + z^6\left(-\frac{1}{6!} + \frac{1}{2!\,4!} - \frac{1}{3(2!)^3}\right) + \ldots \\
&= -\frac{z^2}{2} + z^4\left(\frac{1}{24} - \frac{1}{8}\right) + z^6\left(-\frac{1}{720} + \frac{1}{48} - \frac{1}{24}\right) + \ldots \\
&= -\frac{z^2}{2} - \frac{z^4}{12} + \frac{z^6}{720}(-1 + 15 - 30) + \ldots \\
&= -\frac{z^2}{2} - \frac{z^4}{12} - \frac{z^6}{45} - \ldots
\end{aligned}$$

(c)(ii)

$$h'(z) = \frac{-\sin z}{\cos z} = -\tan z.$$

Therefore using the Differentiation Rule (Unit B3, Section 2, Para. 9) we have

$$\tan z = \frac{d}{dz}\left(\frac{z^2}{2} + \frac{z^4}{12} + \frac{z^6}{45} + \ldots\right) = z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots$$

Question 11

(a)

$f(z) = \dfrac{\pi \cot \pi z}{16(z - i/4)(z + i/4)}$ has simple poles at z = ±i/4.

By the cover-up rule (Unit C1, Section 1, Para. 3)

$$\text{Res}(f, i/4) = \frac{\pi \cot(i\pi/4)}{16(i/4 + i/4)} = \frac{\pi \cot(i\pi/4)}{8i}, \text{ and}$$

$$\text{Res}(f, -i/4) = \frac{\pi \cot(-i\pi/4)}{16(-i/4 - i/4)} = \frac{\pi \cot(-i\pi/4)}{-8i}.$$

Since sin(iz) = i sinh z and cos(iz) = cosh z then cot(iz) = - i coth(z).
(Unit A2, Section 4, Para. 7).

Therefore $\text{Res}(f, i/4) = -\dfrac{\pi \coth(\pi/4)}{8}$ and

$$\text{Res}(f, -i/4) = \frac{\pi \coth(-\pi/4)}{8} = -\frac{\pi \coth(\pi/4)}{8}.$$ (Unit A2, Section 4, Para. 6)

f(z) = g(z) / h(z) where $g(z) = \dfrac{\pi \cos \pi z}{16z^2 + 1}$ and h(z) = sin πz.

g and h are analytic at 0, h(0) = 0, and $h'(0) = \pi \cos(0) = \pi \neq 0$.

Therefore by the g/h rule (Unit C1, Section 1, Para. 2)

$$\text{Res}(f, 0) = \frac{g(0)}{h'(0)} = \frac{\pi * 1}{1} * \frac{1}{\pi} = 1.$$

[You could also use Unit C1, Section 4, Para 1 – last line]

(b)

The method given in Unit C1, Section 4, Para. 1 will be used.

$f(z) = \pi \cot \pi z * \phi(z)$ where $\phi(z) = 1/(16z^2 + 1)$.

ϕ is an even function which is analytic on $\mathbb{C}$ except for simple poles at the non-integral points $z = \pm i/4$.

Let S_N be the square contour with vertices at $(N + ½)(\pm 1 \pm i)$.

On S_N we have $|z| \geq N + ½$ so, using the backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 2),

$$|16z^2 + 1| \geq |\,|16z^2| - 1\,| \geq 16(N + ½)^2 - 1 \geq 16N^2.$$

On S_N we also have $\cot \pi z \leq 2$ (Unit C1, Section 4, Para. 2) so on S_N

$$|\,f(z)\,| \leq \frac{\pi\,(2)}{16N^2}$$

The length of the contour S_N is $4(2N + 1)$.

As f is continuous on the contour S_N then by the Estimation Theorem (Unit B1, Section 4, Para. 3) we have

$$\left|\int_{S_N} f(z)dz\right| \leq \frac{2\pi}{16N^2} 4(2N+1) = \frac{\pi}{2N}\left(2 + \tfrac{1}{N}\right)$$

Hence $\lim_{N \to \infty} \left|\int_{S_N} f(z)dz\right| = 0$.

Therefore the conditions specified in Unit C1, Section 4, Para. 1 hold so

$$\sum_{n=1}^{\infty} \frac{1}{16n^2+1} = -\frac{1}{2}\left(\mathrm{Res}(f,0) + \mathrm{Res}(f,i/4) + \mathrm{Res}(f,-i/4)\right)$$

$$= -\frac{1}{2}\left(1 - \frac{\pi\ \coth(\pi\ /4)}{8} - \frac{\pi\ \coth(\pi\ /4)}{8}\right) = -\frac{1}{2} + \frac{\pi}{8}\coth\frac{\pi}{4}.$$

(c)

$$\sum_{n=-\infty}^{\infty} \frac{1}{16n^2+1} = \sum_{n=-\infty}^{-1} \frac{1}{16n^2+1} + 1 + \sum_{n=1}^{\infty} \frac{1}{16n^2+1}$$

$$= 1 + 2\sum_{n=1}^{\infty} \frac{1}{16n^2+1} = \frac{\pi}{4}\coth\frac{\pi}{4}.$$

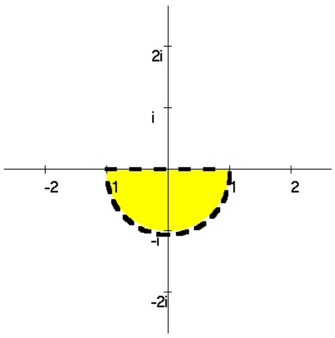
Question 12

(a)

Using the formula for a transformation mapping points to the standard triple (D1, Sect. 2, Para. 11) then the Möbius transformation $\hat{f}_1$ which maps 1, -i , and - 1 to the standard triple of points 0, 1, and ∞ respectively is

$$f_1(z) = \frac{(z-1)}{(z+1)}\frac{(-i+1)}{(-i-1)} = \frac{(z-1)}{(z+1)}\frac{(-i+1)(i-1)}{(-i-1)(i-1)} = \frac{iz-i}{z+1}$$

(b)(i)



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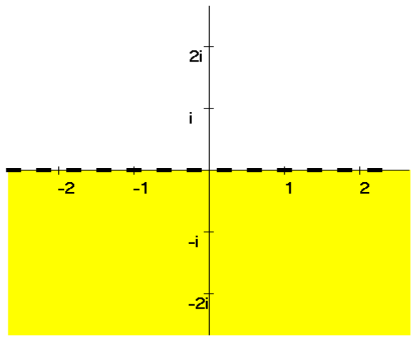
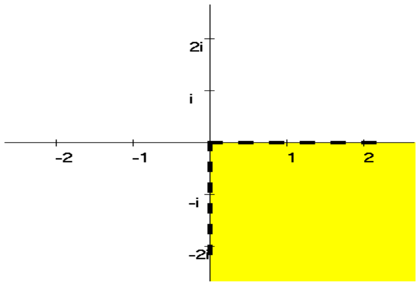


S

(b)(ii) Since $\hat{f}_1$ maps 1 to 0 and -1 to ∞ then the boundaries of R are mapped to extended lines originating at the origin. As the angle between the boundaries meeting at 1 is π/2 and the mapping $\hat{f}_1$ is conformal then the angle between the extended lines at the origin is also π/2.

Since $\hat{f}_1$ maps 0 to -i then the straight boundary is mapped to the negative imaginary-axis. As the other boundary is reached by an anti-clockwise rotation of π/2 the other boundary is the positive real axis.

Therefore the image of R under $\hat{f}_1$ is



(b)(iii) A conformal mapping from f(R) onto T is the power function $w = g(z) = z^2$.

Since the combination of conformal mapping is also conformal then a conformal mapping from R to S is

$$f(z) = (g \circ f_1)(z) = g\left(\frac{iz - i}{z+1}\right) = \left(\frac{iz - i}{z+1}\right)^2.$$

(b)(iv)

Since $f^{-1} = (g \circ f_1)^{-1} = (f_1^{-1} \circ g^{-1})$ then using Unit D1, Section 2, Para. 6 we have

$$f^{-1}(z) = f_1^{-1}\left(\sqrt{z}\right) = \frac{z^{1/2} + i}{-z^{1/2} + i}$$