Question 1

(a) $(2 - 2i)^4 = (-8i)^2 = -64$.

(b) $8i = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

The principal cube root is (Unit A1, Section 3, Para 4)

$$8^{1/3}\left(\cos\left(\frac{1}{3}\left(\frac{\pi}{2}\right)\right) + i\sin\left(\frac{1}{3}\left(\frac{\pi}{2}\right)\right)\right) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i.$$

(c) $\text{Log}\,(1 - i) = \log_e(|1 - i|) + i\text{Arg}(1 - i) = \log_e\sqrt{2} - \frac{\pi}{4}i$ (Unit A2, Section 5, Para. 1)

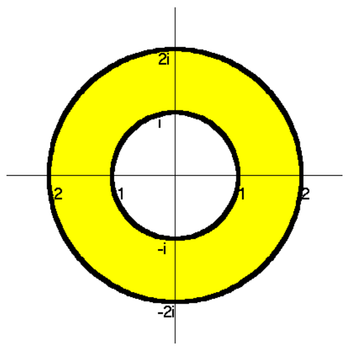
(d) $(-1)^{-i} = \exp(-i\,\text{Log}\,(-1))$ (Unit A2, Section 5, Para. 3)

$= \exp(-i\,\{\log_e|-1| + i\,\text{Arg}\,(-1)\})$ (Unit A2, Section 5, Para. 1)

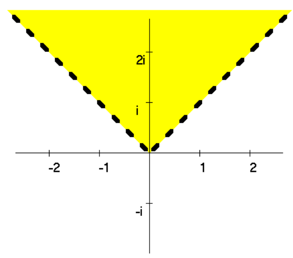
$= \exp(-i\{0 + i\pi\}) = \exp(\pi)$

Question 2

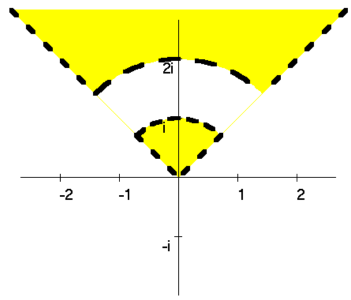
(a)



A



B



C = B - A

(b)(i) A is not a region as it is not open.
B is a region.
C is not a region as it is not connected.

(b)(ii) A is compact.
B and C are not compact as they are neither closed or bounded.

Question 3

(a)

(a)(i) The standard parametrization for the line segment Γ is (Unit A2, Section 2, Para. 3)

$$\gamma(t) = (1 - t)i + 2t \qquad (t \in [0, 1])$$

(a)(ii) $z = (1 - t)i + 2t$, $\text{Re } z = 2t$, $dz = (2 - i)\, dt$.

Since γ is a smooth path then (Unit B1, Section 2, Para. 1)

$$\int_\Gamma \text{Re } z\, dz = \int_0^1 2t(2 - i)dt = (2 - i)\left[t^2\right]_0^1 = 2 - i\,.$$

(b)

The length of Γ is $L = \sqrt{2^2 + 1^2} = \sqrt{5}$.

Using the Triangle Inequality (Unit A1, Section 5, Para. 3b) then, for $z \in \Gamma$, we have

$$|\cos z| = \tfrac{1}{2}\left|e^{iz} + e^{-iz}\right| \le \tfrac{1}{2}\left\{\left|e^{iz}\right| + \left|e^{-iz}\right|\right\} = \tfrac{1}{2}\left\{e^{\text{Re}(iz)} + e^{\text{Re}(-iz)}\right\}$$
$$= \frac{1}{2}\left\{e^{-y} + e^{y}\right\} \le \frac{1}{2}\left\{e^1 + e^1\right\} = e$$

Using the Backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 3c) then, for $z \in \Gamma$, we have

$$|9 + z^2| \ge \left||9| - |z^2|\right| \ge |9 - 4| = 5$$

Therefore $M = \left|\dfrac{\cos z}{9 + z^2}\right| \le \dfrac{e}{5}$ for $z \in \Gamma$.

$f(z) = \dfrac{\cos z}{9 + z^2}$ is continuous on $\mathbb{C} - \{-3i, 3i\}$ and hence on the line Γ.

Therefore by the Estimation Theorem (Unit B1, Section 4, Para. 3)

$$\left|\int_\Gamma f(z)dz\right| \le ML = \frac{e}{5} * \sqrt{5} = \frac{e}{\sqrt{5}}\,.$$

Question 4

(a)

Let $\mathcal{R}$ be the simply-connected region {z: |z| < 2}. C is a closed contour in $\mathcal{R}$, and $f(z) = \dfrac{\exp z}{(3-z)^3}$ is analytic on $\mathcal{R}$.

By Cauchy's Theorem (Unit B2, Section 1, Para. 4)

$$\int_C \frac{\exp z}{(3-z)^3}\,dz = 0 .$$

(b)

Let $\mathcal{R}$ be the simply connected region $\mathbb{C}$. $\mathcal{R}$ is a simply-connected region and C is a simple-closed contour in ***R***. As exp(3-z) is analytic on $\mathcal{R}$ then by Cauchy's Integral Formula (Unit B2, Section 2, Para. 1)

$$\int_C \frac{\exp(3-z)}{z}\,dz = 2\pi i * \exp(3-0) = 2\pi e^3 i .$$

(c)

Let $\mathcal{R}$ be the simply connected region $\mathbb{C}$. $\mathcal{R}$ is a simply-connected region and C is a simple-closed contour in ***R***. As f(z) = exp(3-z) is analytic on $\mathcal{R}$ then by Cauchy's n'th Derivative Formula (Unit B2, Section 3, Para. 1) with n = 2 and $\alpha = 0$ we have

$$\int_C \frac{\exp(3-z)}{z^3}\,dz = \frac{2\pi i}{2!} * f^{(2)}(0) .$$

$f^{(1)}(z) = -\exp(3-z)$ and $f^{(2)}(z) = \exp(3-z)$ so $\displaystyle\int_C \frac{\exp(3-z)}{z^3}\,dz = \pi e^3 i$

Question 5

(a)

$z^3 = -1 = e^{i\pi}$. Therefore $z^3 + 1$ has zeros at $z = e^{\pi i/3}$, $e^{\pi i} = -1$, and $e^{5\pi i/3} = e^{-\pi i/3}$.
Therefore f has simple poles at these points.

Let $g(z) = 1$ and $h(z) = z^3 + 1$. Then $h'(z) = 3z^2$.

If α is one of the poles then g and h are analytic at α, $h(\alpha) = 0$, and $h'(\alpha) = 3\alpha^2 \neq 0$.
Therefore by the g/h rule (Unit C1, Section 1, Para. 2)

$$\text{Res}\left(f, e^{\pi i/3}\right) = \frac{1}{3e^{2\pi i/3}} = \frac{1}{3}e^{-2\pi i/3}$$

$$\text{Res}(f, -1) = \frac{1}{3}$$

$$\text{Res}\left(f, e^{-\pi i/3}\right) = \frac{1}{3e^{-2\pi i/3}} = \frac{1}{3}e^{2\pi i/3}$$

(b)

I shall use the result given in Unit C1, Section 3, Para. 8.

Let $p(t) = 1$, $q(t) = t^3 + 1$.

p and q are polynomial functions such that the degree of q exceeds that of p by at least 2, and the pole of p/q on the real axis is simple. Therefore

$$\int_{-\infty}^{\infty} \frac{1}{t^3+1}\,dt = 2\pi i S + \pi i T$$

where S is the sum of the residues of f at the poles in the upper half-plane, and
T is the sum of the residues of f at the poles on the real axis.

As $S = \text{Res}(f, e^{i\pi/3})$ and $T = \text{Res}(f, -1)$.

$$\int_{-\infty}^{\infty} \frac{1}{t^3+1}\,dt = 2\pi i\left(\frac{e^{-2\pi i/3}}{3}\right) + \pi i\left(\frac{1}{3}\right)$$

$$= \frac{2\pi i}{3}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) + \frac{\pi i}{3} = \frac{\sqrt{3}\pi}{3}$$

[[As it is a real integral we expect the imaginary terms to cancel]]

Question 6

(a)(i) Let $g_1(z) = z^5$.

For $z \in C_1$ then, using the Triangle Inequality (Unit A1, Section 5, Para. 3),

$$| f(z) - g_1(z) | = |-3z^3 + i| \leq |-3z^3| + |i| = 24 + 1 < 32 = | g_1(z) |.$$

As f is a polynomial then it is analytic on the simply-connected region $\boldsymbol{R} = \mathbb{C}$. Since C_1 is a simple-closed contour in $\boldsymbol{R}$ then by Rouch■ 's theorem (Unit C2, Section 2, Para. 4) f has the same number of zeros as g_1 inside the contour C_1. Therefore f has 5 zeros inside C_1.

(a)(ii) Let $g_2(z) = -3z^3$.
On the contour C_2 we have, using the Triangle Inequality,

$$| f(z) - g_2(z) | = |z^5 + i| \leq |z^5| + |i| = 1 + 1 < 3 = | g_2(z) |.$$

As C_2 is a simple-closed contour in $\boldsymbol{R}$ then by Rouch■'s theorem f has the same number of zeros as g_2 inside the contour C_2. Therefore f has 3 zeros inside C_2.

(b)

From part(a) f(z) has 2 solutions in the set $\{z: 1 \leq z < 2 \}$. Therefore we have to find if there are any solutions on C_2.

From part (a), on C_2 we have $|z^5 + i | \leq 2$.

Therefore, using the Backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 3c)

$$|f(z)| \geq | |-3z^3| - |z^5 + i|| \geq | 3 - 2| =1, \text{ on } C_2.$$

As f(z) is non-zero on C_2 then there are exactly 2 solutions in the set $\{z: 1 < z < 2 \}$.

Question 7

(a)

The conjugate velocity function $\overline{q}(z) = 1/z^2$.
As q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C} - \{0\}$ and $\overline{q}$ is analytic on $\mathbb{C} - \{0\}$ then q is a model fluid flow (Unit D2, Section 1, Para. 14).

(b) On $\mathbb{C} - \{0\}$, $\Omega(z) = -\frac{1}{z}$ is a primitive of $\overline{q}$. Therefore Ω is a complex potential function for the flow (Unit D2, Section 2, Para. 1).

The stream function $\Psi(x, y) = \operatorname{Im}\Omega(z)$ (Unit D2, Section 2, Para. 4)

$$= \operatorname{Im}\left(-\frac{1}{x + iy}\right)$$, where $z = x + iy$, $(x,y) \neq (0, 0)$

$$= \operatorname{Im}\left(-\frac{x - iy}{x^2 + y^2}\right) = \frac{y}{x^2 + y^2}$$

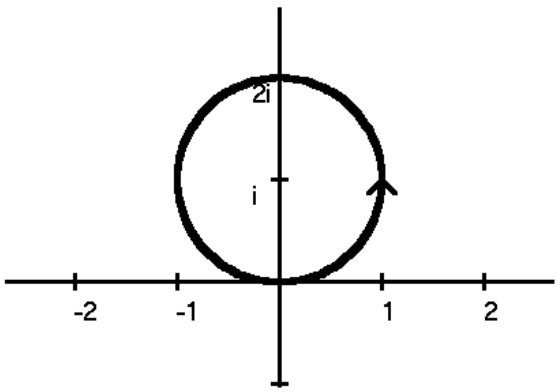
A streamline through the point 2i satisfies the equation

$$\frac{y}{x^2 + y^2} = \Psi(0,2) = \tfrac{1}{2}$$ (Unit D2, Section 2, Para. 4)

Therefore the streamline through 2i has the equation $x^2 + y^2 - 2y = 0$ or

$$x^2 + (y - 1)^2 = 1$$

Since q(2i) = -1/4 (-x direction) then the direction of flow is as shown.



(c)

The flux of q across the unit circle $C = \{z : |z| = 1\}$ is (Unit D2, Section 1, Para. 10)

$$\operatorname{Im}\left(\int_C \overline{q}(z)dz\right) = \operatorname{Im}\left(\int_C \frac{1}{z^2} dz\right) = 0$$ by Cauchy's Residue Theorem or the n^{th} Derivative formula (Unit B4, Section 3, Para. 1).

Question 8

(a)

If α is a fixed point then $f(\alpha) = \alpha^2 - 2\alpha + 2 = \alpha$ (Unit D3, Sect. 1, Para 3).
As $\alpha^2 - 3\alpha + 2 = (\alpha - 1)(\alpha - 2) = 0$ then there are fixed points at 1 and 2.

$f'(z) = 2z - 2$.

As $f'(1) = 0$ then 1 is a super-attracting fixed point (Unit D3, Sect. 1, Para. 5).
As $f'(2) = 2$ then 2 is a repelling fixed point.

(b)(i) [[From the diagram in Handbook looks as if point not in Mandelbrot set.]]

$P_c(0) = -\frac{1}{2}(3+i)$.

$P_c^2(0) = \frac{1}{4}(3+i)^2 - \frac{1}{2}(3+i) = \left(2+\frac{3}{2}i\right) - \frac{1}{2}(3+i) = \frac{1}{2}+i$.

$P_c^3(0) = \left(\frac{1}{2}+i\right)^2 - \frac{1}{2}(3+i) = \left(-\frac{3}{4}+i\right) - \frac{1}{2}(3+i) = -\frac{9}{4}+\frac{1}{2}i$.

As $|P_c^3(0)| > 2$ then c does not lie in the Mandelbrot set (Unit D3, Section 4, Para. 5).

(b)(ii)

Since $|c+1| = \left|\frac{1}{6}i\right| < \frac{1}{4}$ then P_c has an attracting 2-cycle (Unit D3, Section 4, Para. 9).
Therefore c belongs to the Mandelbrot set (Unit D3, Section 4, Para. 8).

Question 9

(a)

(a)(i)

$f(z) = z\,\mathrm{Re}\,z + |z|^2 = (x + iy)x + (x^2 + y^2) = u(x, y) + iv(x, y)$,
where $u(x,y) = 2x^2 + y^2$, and $v(x,y) = xy$.

(a)(ii)

$\frac{\partial u}{\partial x}(x, y) = 4x$, $\frac{\partial u}{\partial y}(x, y) = 2y$, $\frac{\partial v}{\partial x}(x, y) = y$, $\frac{\partial v}{\partial y}(x, y) = x$

If f is differentiable then the Cauchy-Riemann equations hold (Unit A4, Section 2, Para. 1). If they hold at (a, b)

$\frac{\partial u}{\partial x}(a, b) = 4a = a = \frac{\partial v}{\partial y}(a, b)$, and

$\frac{\partial v}{\partial x}(a, b) = b = -2b = -\frac{\partial u}{\partial y}(a, b)$

Therefore the Cauchy-Riemann equations only hold at (0, 0).

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous at (0, 0).
3. satisfy the Cauchy-Riemann equations at (0, 0)

then, by the Cauchy-Riemann Converse Theorem (Unit A4, Section 2, Para. 3), f is differentiable at 0.

As the Cauchy-Riemann only hold at (0, 0) then f is not differentiable on any region surrounding 0. Therefore f is not analytic at 0. (Unit A4, Section 1, Para. 3)

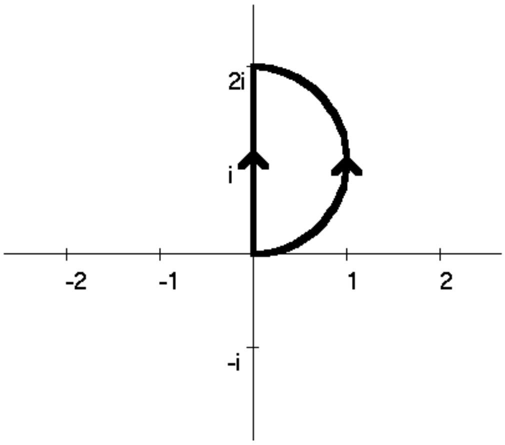
(a)(iii)

$f'(0, 0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 0$ (Unit A4, Section 2, Para. 3).

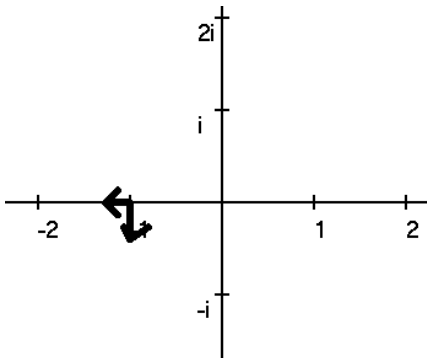
(b)

(i) Since g is a polynomial then g is entire (Unit A4, Section 1, Para. 7) and $g'(z) = 2z$ on $\mathbb{C}$. As $g(z) \neq 0$ when $z \neq 0$ then g is conformal on $\mathbb{C} - \{0\}$ (Unit A4, Section 4, Para. 6).

(ii) As g is analytic on $\mathbb{C}$ and $g'(2i) \neq 0$ then a small disc centred at 2i is mapped approximately (Unit A4, Section 1, Para. 11) to a small disc centred at $g(2i) = -4 + 3 = -1$. The disc is rotated by $\text{Arg}\,(g'(2i)) = \text{Arg}\,4i = \pi/2$, and scaled by a factor $|g'(2i)| = 4$.



(iv)



(v) $(g \circ \gamma_i)'(t) = \gamma_i'(t) g'(\gamma_i(t))$ for i = 1, 2.

As $g'(z) = 2z$ then when $\gamma_i(t) = 0$, $(g \circ \gamma_i)'(0) = 0$ (i = 1, 2).

As the paths at z = 0 are not at right-angles then g is not conformal at 0.

Question 10

(a)

(a)(i) f has simple poles at z = 0 and z = 3.

(a)(ii) $$f(z) = \frac{9}{z(z-3)} = -\frac{3}{z\left(1-\frac{z}{3}\right)}$$

$$= -\frac{3}{z}\left\{\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right\}$$

since |z / 3| < 1 on {z : 0 < |z| < 3} (Unit B3, Section 3, Para. 5)

Hence the required Laurent series about 0 is

$$-\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{n-1} = -\frac{3}{z} - 1 - \frac{z}{3} - \frac{z^2}{9} - \ldots - \left(\frac{z}{3}\right)^{n-1} - \ldots .$$

(a)(iii) $$f(z) = \frac{9}{z(z-3)} = \frac{9}{(z-3)}\frac{1}{(z-3)+3} = \frac{9}{(z-3)^2}\frac{1}{1+\frac{3}{z-3}}$$

$$= \frac{9}{(z-3)^2}\left\{\sum_{n=0}^{\infty}\left(\frac{-3}{z-3}\right)^n\right\}$$

since |3 / (z-3)| < 1 on {z : |z - 3| > 3} (Unit B3, Section 3, Para. 5)

Therefore the required Laurent series about 3 is

$$\sum_{n=0}^{\infty}\left(\frac{-3}{z-3}\right)^{n+2} = \frac{9}{(z-3)^2} - \frac{27}{(z-3)^3} + \frac{81}{(z-3)^4} - \ldots + \left(\frac{-3}{z-3}\right)^{n+2} - \ldots .$$

(b)

(b)(i) By the Composition Rule (Unit B3, Section 4, Para. 3) the Taylor series for g about 0 on $\mathbb{C}$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left\{2\sum_{m=0}^{\infty}(-1)^m \frac{z^{2m+1}}{(2m+1)!}\right\}^{2n}$$

$$= 1 - \frac{2^2}{2!}\left\{z - \frac{z^3}{3!} + \ldots\right\}^2 + \frac{2^4}{4!}\left\{z - \frac{z^3}{3!} + \ldots\right\}^4 - \ldots$$

$$= 1 - 2\left\{z^2 - \frac{z^4}{3} + \ldots\right\} + \left\{\frac{2}{3}z^4 - \ldots\right\} = 1 - 2z^2 + \frac{4}{3}z^4 - \ldots \quad \text{up to the term in } z^4.$$

Since g is analytic on $\mathbb{C}$ then by Taylor's Theorem (Unit B3, Section 3, Para. 1) then the representation of g is unique on all open discs centred at 0 in the sense that if

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

then the coefficients a_n are those found above.

Let f(z) = g(1/z). f is analytic on the punctured disc $\mathbb{C}$ - {0} which contains the circle C centred at 0.

The Laurent series about 0 for f on this disc is

$$1 - 2\frac{1}{z^2} + \frac{4}{3}\frac{1}{z^4} - \ldots = \sum_{n=-\infty}^{\infty} a_n z^n$$

using Laurent's Theorem (Unit B4, Section 2, Para. 5) then

$$\int_C zg(1/z)dz = \int_C \frac{f(w)}{w^{-1}}dw = 2\pi i a_{-2} = -4\pi i$$

and $\int_C z^2 g(1/z)dz = \int_C \frac{f(w)}{w^{-2}}dw = 2\pi i a_{-3} = 0$.

Question 11

(a)(i)

Putting $z = x + iy$ where $x, y \in \mathbf{R}$ then

$\exp(iz) = \exp(ix - y) = e^{-y}(\cos x + i \sin x)$

Since $|\exp z| = e^{\mathrm{Re}\, z}$ (Unit A2, Section 4, Para. 2b) then

$|\exp(iz)| = \exp(e^{-y}\cos x)$

(a)(ii)

Let $f(z) = \exp(e^{iz})$ and $R = \{z : -\pi \le \mathrm{Re}\, z \le \pi,\ -1 \le \mathrm{Im}\, z \le 1\}$.

As f is analytic on the bounded region R and continuous on $\overline{R}$ then by the Maximum Principle (Unit C2, Section 4, Para. 4) there exists an $\alpha \in \partial R$ such that $|f(z)| \le |f(\alpha)|$ for $z \in \overline{R}$.

From part (i) we have $|\exp(iz)| = \exp(e^{-y}\cos x)$.

As $e^{-y}\cos x$ is real and exp is a monotonic function for real values we need to find the maximum of $e^{-y}\cos x$ on ∂R. e^{-y} is a maximum when $y = -1$ and $\cos x$ is a maximum when $x = 0$. These values can be attained simultaneously on ∂R.

Therefore $\max\{\exp(e^{iz}) : -\pi \le \mathrm{Re}\, z \le \pi, -1 \le \mathrm{Im}\, z \le 1\} = e^{e}$.

The maximum only occurs when $z = -i$ as at all other points in $\overline{R}$ either

$e^{y} < e^{1}$ or $\cos x < 1$.

(b)

Let $D_f = \{z: |z| < 4\}$ and $D_g = \{z: |z| > 4\}$.
Since $D_f \cap D_g = \varnothing$ then f and g are not direct analytic continuations of each other.

Let $h(z) = \dfrac{4}{4-z}$ on D_h, where $D_h = \mathbf{C} - \{4\}$.

When $z \in D_f$ then $|z|/4 < 1$ and the geometric series $\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n$ is convergent and has the sum

$$\frac{1}{1-\frac{z}{4}} = \frac{4}{4-z}. \qquad \text{(Unit B3, Section 3, Para. 5)}$$

Since f = h when $z \in D_f \subseteq D_f \cap D_h$ then h is an analytic continuation of f.

When $z \in D_g$ then $4/|z| < 1$ and the geometric series $\sum_{n=0}^{\infty}\left(\frac{4}{z}\right)^n$ is convergent and has the sum

$$\frac{1}{1-\frac{4}{z}} = \frac{z}{z-4}.$$

Therefore $-\sum_{n=1}^{\infty}\left(\frac{4}{z}\right)^n = -\frac{4}{z}\sum_{n=0}^{\infty}\left(\frac{4}{z}\right)^n = \frac{4}{z-4}$ when $z \in D_g$.

Since g = h when $z \in D_g \subseteq D_g \cap D_h$ then g is an analytic continuation of h.

Since (f, D_f), (g, D_g), (h, D_h) form a chain then f and g are indirect analytic continuations of each other.

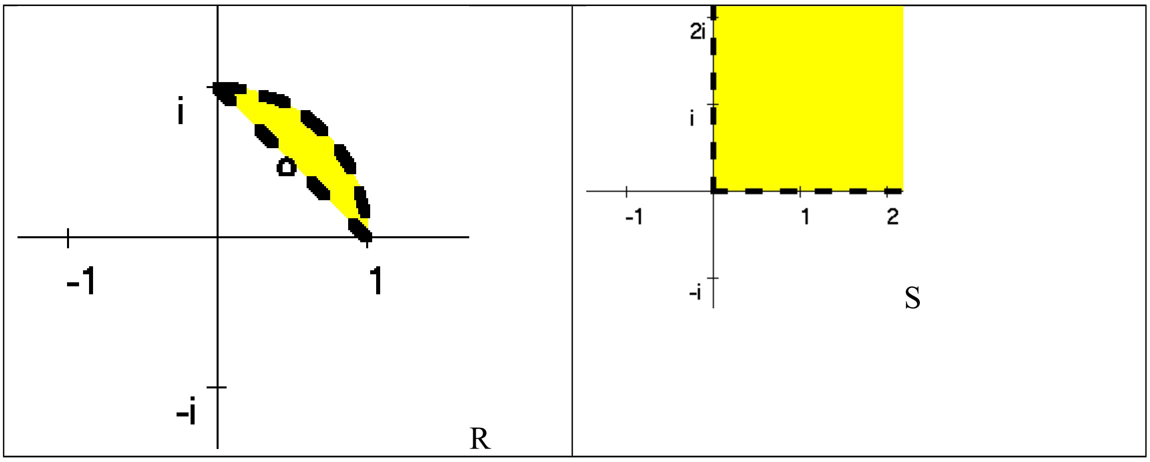
Question 12

(a)

Using the formula for a transformation mapping points to the standard triple (Unit D1, Section 2, Para. 11) then the M★bius transformation $\hat{f}_1$ which maps i, ½(i + 1) , and 1 to 0, 1, and ∞ respectively is

$$f_1(z) = \frac{(z-i)}{(z-1)}\frac{\left(\frac{1}{2}(1+i)-1\right)}{\left(\frac{1}{2}(1+i)-i\right)} = \frac{(z-i)}{(z-1)}\frac{\frac{1}{2}(-1+i)}{\frac{1}{2}(1-i)} = \frac{-z+i}{z-1}$$

(b)(i)

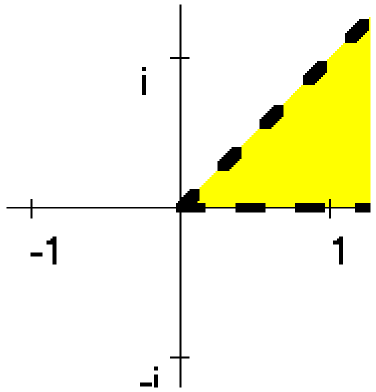


(b)(ii) Since $\hat{f}_1$ maps i to 0 and 1 to ∞ then the straight and curved boundaries of R are mapped to extended lines originating at the origin.

From part (a), ½(1 +i) is mapped to a point on the positive real-axis then the straight boundary is mapped to the non-negative x-axis.

At z = i the angle between the boundary lines of R are at an angle of π/4. Therefore as the transformation is conformal then this is also the angle at the origin of the transformed lines. Going along the straight line boundary in R from i towards 1 the region to be mapped is on the left. Therefore the image of the region is above the non-negative real axis.

Therefore the image of R under $\hat{f}_1$ is $R_1 = \{z \in \mathbb{C}: 0 < \text{Arg } z < \pi/4\}$



(b)(iii) A conformal mapping from R_1 onto S is the power function $w = g(z) = z^2$. Since the combination of conformal mapping is also conformal then a conformal mapping from R to S is

$$f(z) = \left(\frac{-z+i}{z-1}\right)^2$$

(b)(iv)

Since $f^{-1} = (g \circ f_1)^{-1} = (f_1^{-1} \circ g^{-1})$ then using Unit D1, Section 2, Para. 6 we have

$$f^{-1}(z) = \frac{z^{1/2}+i}{z^{1/2}+1}$$