Question 1

(a) 2 marks

(a)(i) $|\alpha| = \sqrt{(-2)^2+2^2}=\sqrt{8}=2\sqrt{2}$ (Unit A1, Section 2, Para. 2)

(a)(ii) Arg $\alpha = 3\pi/4$. (Unit A1, Section 2, Para. 8)

(b) 6 marks

(b)(i) $\alpha=2\sqrt{2}\exp(3i\pi/4)$

$$\frac{1}{\alpha}=\frac{1}{2\sqrt{2}}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)=-\frac{1}{4}-i\frac{1}{4}$$ (Unit A1, Section 2, Para. 12)

(b)(ii) The principal value of $\alpha^{1/3}$ is (Unit A1, Section 3, Para 4)

$$(2\sqrt{2})^{1/3}\left(\cos\left(\frac{1}{3}\left(\frac{3\pi}{4}\right)\right)+i\sin\left(\frac{1}{3}\left(\frac{3\pi}{4}\right)\right)\right)$$

$$=\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)=1+i$$

(b)(iii) Log $\alpha = \log_e(2\sqrt{2})+i(3\pi/4)=\frac{3}{2}\log_e 2+\frac{3\pi}{4}i$ (Unit A2, Section 5, Para. 1)

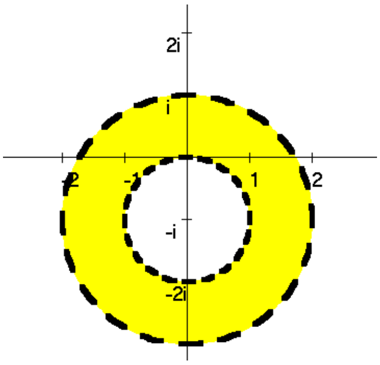
(b)(iv) Arg$(\alpha^3) = \frac{1}{4}\pi$ as $\frac{9}{4}\pi=\frac{1}{4}\pi$.

Therefore Log(α^3) = 3 Log α - $2\pi i$ $=\frac{9}{2}\log_e 2+\frac{\pi}{4}i$

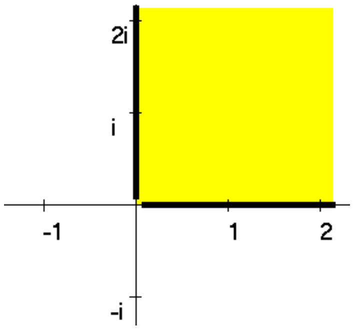
(Unit A2, Section 5, Paras. 1 & 2)

Question 2

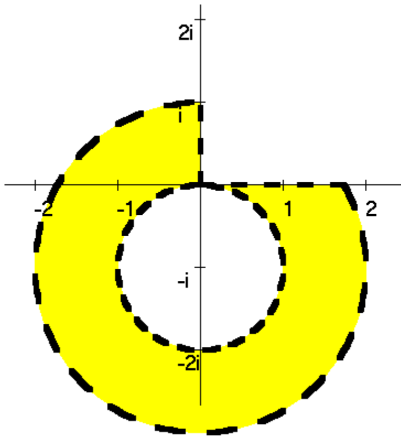
(a) 3 marks



A



B



C = A - B

Note origin not included in B as Arg not defined there. Also origin not in C.

(b) 4 marks

(b)(i) A and C.
(b)(ii) C.
(b)(iii) B.

(c) 1 mark

{0, 1}.

Question 3

(a) 3 marks

(a)(i) The standard parametrization for the circle Γ is (Unit A2, Section 2, Para. 3)

$$\gamma(t) = 2(\cos t + i\sin t) = 2e^{it} \qquad (t \in [0, 2\pi])$$

(a)(ii) $\gamma'(t) = 2ie^{it}$

Since γ is a smooth path then (Unit B1, Section 2, Para. 1)

$$\int_\Gamma \bar{z}\,dz = \int_0^{2\pi} \overline{\gamma(t)}\gamma'(t)\,dt = \int_0^{2\pi} 2e^{-it}(2ie^{it})\,dt = 4\mathrm{i}\int_0^{2\pi} dt = 8\pi\mathrm{i}$$

(b) 5 marks

The length of Γ is L = 2π * 2 = 4π.

Using the Triangle Inequality (Unit A1, Section 5, Para. 3b) then, for z ∈ Γ, we have

$$|\bar{z}^2 - 1| \le |\bar{z}^2| + 1 = |z|^2 + 1 = 4 + 1 = 5$$

Using the Backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 3c) then, for z ∈ Γ, we have

$$|z^2 - 1| \ge ||z^2| - 1| = |4 - 1| = 3$$

Therefore $M = \left|\frac{\bar{z}^2 - 1}{z^2 - 1}\right| \le \frac{5}{3}$ for z ∈ Γ.

$f(z) = \frac{\bar{z}^2 - 1}{z^2 - 1}$ is continuous on $\mathbb{C}$ – {-1, 1} and hence on the circle Γ.

Therefore by the Estimation Theorem (Unit B1, Section 4, Para. 3)

$$\left|\int_\Gamma f(z)\,dz\right| \le ML = \frac{5}{3} * 4\pi = \frac{20}{3}\pi$$

Question 4

(a) 3 marks

$\mathbb{C}$ is a simply-connected region, C is a simple-closed contour in $\mathbb{C}$, and
f(z) = exp (iπz) is analytic on $\mathbb{C}$.

As -1 lies inside the circle C then by Cauchy's Integral formula (Unit B2, Section 1, Para. 4) then

$$\int_C \frac{e^{i\pi z}}{z+1}\,dz = 2\pi \mathrm{i}\, f(-1) = 2\pi \mathrm{i} * e^{-i\pi} = -2\pi \mathrm{i}$$

(b) 2 marks

Let $\boldsymbol{R} = \{z \in C : |z - i| < 5^{1/2}\}$. $\boldsymbol{R}$ is a simply-connected region and C is a simple-closed contour in $\boldsymbol{R}$. As $\frac{e^{i\pi z}}{z+3}$ is analytic on $\boldsymbol{R}$ then by Cauchy's Theorem (Unit B2, Section 1, Para. 4)

$$\int_C \frac{e^{i\pi z}}{z+3}\,dz = 0$$

(c) 3 marks Unit B2

Let g(z) = sin(z- π/2). g is a function which is analytic on the simply-connected region $\mathbb{C}$ (Unit B2, Section 1, Para. 3).

The contour C is a simple-closed contour in $\mathbb{C}$. Since z^3 is zero inside the circle C then using Cauchy's n^{th} Derivative Formula (Unit B2, Section 3, Para. 1), with n = 2 and α = 0 we have

$$\int_C \frac{\sin(z-\pi/2)}{z^3}\,dz = \int_C \frac{g(z)}{z^3}\,dz = \frac{2\pi \mathrm{i}}{2!}\,g^{(2)}(0)$$

$$g'(z) = \cos(z-\pi/2)\ .$$

$$g''(z) = -\sin(z-\pi/2)\ .$$

So $g''(0) = -\sin(-\pi/2) = 1$.

Hence $\int_C \frac{\sin(z-\pi/2)}{z^3}\,dz = \pi i$.

Question 5

(a) 3 marks

f is an analytic function with simple poles at z = 0, ½, and 2. Using the cover-up rule (Unit C1, Section 1, Para. 3).

$$\operatorname{Res}(f,0)=\frac{1}{\left(-\frac{1}{2}\right)(-2)}=1.$$

$$\operatorname{Res}\left(f,\tfrac{1}{2}\right)=\frac{\frac{1}{4}+1}{\frac{1}{2}\left(-\frac{3}{2}\right)}=-\frac{5}{3}.$$

$$\operatorname{Res}(f,2)=\frac{4+1}{2\left(\frac{3}{2}\right)}=\frac{5}{3}.$$

(b) 5 marks

I shall use the strategy given in Unit C1, Section 2, Para. 2.

$$\int_0^{2\pi}\frac{\cos t}{5-4\cos t}\,dt=\int_C\frac{\frac{1}{2}\left(z+z^{-1}\right)}{5-4\left(\frac{1}{2}\right)\left(z+z^{-1}\right)}\frac{1}{iz}\,dz$$, where C is the unit circle {z : |z| = 1}.

$$=-\frac{i}{2}\int_C\frac{z^2+1}{z\left(5z-2z^2-2\right)}\,dz$$

$$=\frac{i}{4}\int_C\frac{z^2+1}{z\left(z^2-\frac{5}{2}z+1\right)}\,dz=\frac{i}{4}\int_C\frac{z^2+1}{z\left(z-\frac{1}{2}\right)(z-2)}\,dz$$

f is analytic on the simply-connected region $\mathbb{C}$ except for a finite number of singularities. C is a simple contour in $\mathbb{C}$ not passing through any of the singularities. Since the singularities at z = ½, and 0 are inside the circle C then by Cauchy's Residue Theorem (Unit C1, Section 2, Para. 1) we have

$$\int_0^{2\pi}\frac{\cos t}{5-4\cos t}\,dt=\frac{i}{4}*2\pi i\left\{\operatorname{Res}\ (f,0)+\ \operatorname{Res}\left(f,\tfrac{1}{2}\right)\right\}$$

$$=-\frac{\pi}{2}\left\{1-\frac{5}{3}\right\}=\frac{\pi}{3}$$

Question 6

(a) 7 marks

(a)(i) Let $f(z) = 2z^3 + 5z - 1$ and $g_1(z) = 2z^3$.

For $z \in C_1$ then, using the Triangle Inequality (Unit A1, Section 5, Para. 3),

$$| f(z) - g_1(z) | = |5z - 1| \leq |5z| + |-1| = 11 < 16 = | g_1(z) |.$$

As f is a polynomial then it is analytic on the simply-connected region $\boldsymbol{R} = \mathbb{C}$. Since C_1 is a simple-closed contour in $\boldsymbol{R}$ then by Rouch■ ’s theorem (Unit C2, Section 2, Para. 4) f has the same number of zeros as g_1 inside the contour C_1. Therefore f has 3 zeros inside C_1.

(a)(ii) Let $g_2(z) = 5z$.
On the contour C_2 we have, using the Triangle Inequality,

$$| f(z) - g_2(z) | = |2z^3 - 1| \leq |2z^3| + |-1| = 3$$
$$< 5 = | g_2(z) |.$$

As C_2 is a simple-closed contour in $\boldsymbol{R}$ then by Rouch■ ’s theorem f has the same number of zeros as g_2 inside the contour C_2. Therefore f has 1 zero inside C_2.

(b) 1 mark

$f(z) = 0$ is a polynomial equation with real coefficients. Therefore if α is a solution then so is the complex conjugate $\bar{\alpha}$. If α is the only solution inside C_2 then we must have $\alpha = \bar{\alpha}$. Hence the solution is real.

Clearly α is non-zero. If $\alpha < 0$ then all the terms in $2\alpha^3 + 5\alpha - 1$ are negative so $\alpha < 0$ cannot be a solution. Therefore the solution inside C_2 is real and positive.

Question 7

(a) 1 mark

q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C}$ and the conjugate velocity function $\overline{q}(z) = -iz$ is analytic on $\mathbb{C}$. Therefore q is a model fluid flow on $\mathbb{C}$ (Unit D2, Section 1, Para. 14).

(b) 5 marks

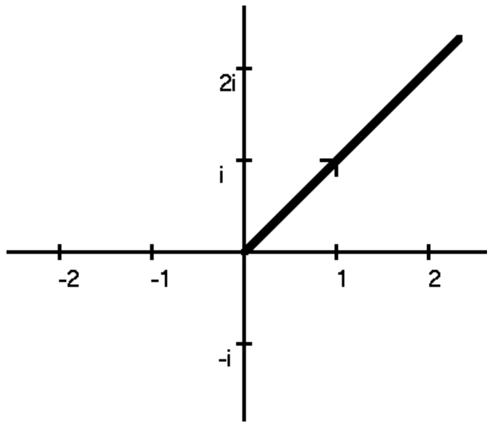
The complex potential function Ω is a primitive of $\overline{q}(z)$ (Unit D2, Section 2, Para. 1). Therefore the complex potential function $\Omega(z) = -iz^2/2$ and the stream function

$$\Psi(x, y) = \operatorname{Im}\Omega(z) \quad \text{(Unit D2, Section 2, Para. 4)}$$

$$= \operatorname{Im}\left(-\tfrac{i}{2}(x+iy)^2\right) \quad \text{, where } z = x + iy$$

$$= \operatorname{Im}\left(-\tfrac{i}{2}(x^2 - y^2 + 2ixy)\right) = \tfrac{1}{2}(-x^2 + y^2)$$

A streamline through 1 + i is given by $\frac{1}{2}(-x^2 + y^2) = \Psi(1,1) = 0$.
Since the streamline goes through 1 + i it must have the equation y = x.
At 1 + i the velocity function q(1 + i) = i(1 - i) = 1 + i (north-east)



(c) 2 marks

Since Γ follows the streamline through 1 + i then the flux of q across Γ is 0 (Unit D2, Section 2, Para. 5).

Question 8

(a) 3 marks

Using the result in Unit D3, Section 2, Para. 1 then the iteration sequence $z_{n+1} = z_n^2 + 6z_n + 5$ is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + (1*5 + 6/2 - 6^2/4) = w_n^2 - 1$$

and conjugating function $h(z) = z + 3$.

Therefore $w_0 = h(z_0) = z_0 + 3 = -3 + 3 = 0$. (Unit D3, Section 1, Para. 7).

(b) 3 marks

If α is a fixed point of P_{-1} (Unit D3, Section 1, Para. 3) then

$$P_{-1}(\alpha) = \alpha^2 - 1 = \alpha.$$

The solutions of $\alpha^2 - \alpha - 1 = 0$ are $\dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$.

$P_{-1}{}'(z) = 2z$.

When $z = \dfrac{1 \pm \sqrt{5}}{2}$ then $|\, P_{-1}{}'(z) \,| = |\, 1 \pm \sqrt{5} \,| > 1$.

Therefore $\dfrac{1 \pm \sqrt{5}}{2}$ are repelling fixed points (Unit D3 Section 1, Para. 5).

(c) 2 marks

Let $c = \frac{1}{2} - i$.

$$P_c(0) = \tfrac{1}{2} - i.$$
$$P_c^2(0) = \left(\tfrac{1}{2} - i\right)^2 + \left(\tfrac{1}{2} - i\right) = \left(\tfrac{1}{4} - 1 - i\right) + \left(\tfrac{1}{2} - i\right) = -\tfrac{1}{4} - 2\mathrm{i}.$$

As $|P_c^2(0)| > 2$ then c does not lie in the Mandelbrot set (Unit D3, Section 4, Para. 5).

Question 9

(a) 8 marks

(a)(i)

$f(z)=\bar{z}+|z|^2=(x-iy)+(x^2+y^2)=u(x,y)+iv(x,y)$,
where u(x,y) = x + x^2 + y^2, and v(x,y) = -y.

(a)(ii)

$\frac{\partial u}{\partial x}(x,y)=1+2\text{x}$, $\frac{\partial u}{\partial y}(x,y)=2\text{y}$, $\frac{\partial v}{\partial x}(x,y)=0$, $\frac{\partial v}{\partial y}(x,y)=-1$

If f is differentiable then the Cauchy-Riemann equations hold (Unit A4, Section 2, Para. 1). If they hold at (a, b)

$\frac{\partial u}{\partial x}(a,b)=1+2\text{a}=-1=\frac{\partial v}{\partial y}(a,b)$, and

$\frac{\partial v}{\partial x}(a,b)=0=-2\text{b}=-\frac{\partial u}{\partial y}(a,b)$

Therefore the Cauchy-Riemann equations only hold at (-1, 0).

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous at (-1, 0).
3. satisfy the Cauchy-Riemann equations at (-1, 0)

then, by the Cauchy-Riemann Converse Theorem (Unit A4, Section 2, Para. 3), f is differentiable at -1.

As the Cauchy-Riemann only hold at (-1, 0) then f is not differentiable on any region surrounding 0. Therefore f is not analytic at -1. (Unit A4, Section 1, Para. 3)

(a)(iii)

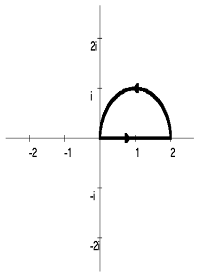
f' (-1, 0) = $\frac{\partial u}{\partial x}(-1,0)+i\frac{\partial v}{\partial x}(-1,0)=-1$ (Unit A4, Section 2, Para. 3).

(b) 10 marks

(i) The domain of g is $\mathbb{C}$ (Unit A4, Section 1, Para. 7) and its derivative $g'(z)=3iz^2$ also has domain $\mathbb{C}$ (Unit A4, Section 3, Para. 4) . Therefore g is analytic on $\mathbb{C}$ - {0}. Since $g'(z) \neq 0$ on $\mathbb{C}$ - {0}then g is conformal on $\mathbb{C}$ - {0} (Unit A4, Section 4, Para. 6).

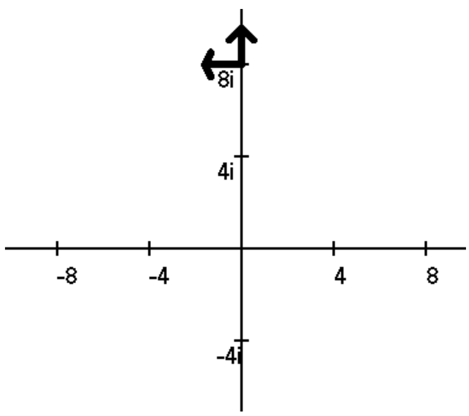
(ii) As g is analytic on $\mathbb{C}$ and $g'(2) \neq 0$ then a small disc centred at 2 is mapped approximately (Unit A4, Section 1, Para. 11) to a small disc centred at $g(2) = 8i$. The disc is rotated by Arg $(g'(2))$ = Arg $12i = \pi/2$, and scaled by a factor $|g'(2)| = 12$.

(iii)



(iv)

The vertical line in the diagram below is $g(\Gamma_1)$. (Unit A4, Section 4, Para. 4)



(v) $(g_\circ\gamma_1)'(t) = g'(\gamma_1(t))\, \gamma_1'(t) = 3i(2t)^2\, 2 = 24it^2$.
$(g_\circ\gamma_2)'(t) = g'(\gamma_2(t))\, \gamma_2'(t) = 3i(1+e^{it})^2\, ie^{it}$.

Since $\gamma_1(0) = 0$ and $\gamma_2(\pi) = 0$ then the slopes of $g(\Gamma_1)$ and $g(\Gamma_2)$ at $g(0)$ are both 0. As Γ_1 and Γ_2 are at right angles at 0 then g is not conformal at 0.

Question 10

(a) 10 marks

(a)(i) f has singularities at z = 0 and z = i. As $\lim_{z \to 0} (z-0) f(z) = 2\mathrm{i}$ and $\lim_{z \to i} (z-i) f(z) = -2\mathrm{i}$ then these are simple poles.

(a)(ii) $f(z) = \dfrac{2}{z(z-i)} = \dfrac{2\mathrm{i}}{z(1+iz)}$

$$= \frac{2\mathrm{i}}{z}\left\{\sum_{n=0}^{\infty} (-iz)^n\right\}$$

since |iz | < 1 on {z : 0 < |z| < 1} (Unit B3, Section 3, Para. 5)

Hence the required Laurent series about 0 is

$$2\sum_{n=0}^{\infty} (-iz)^{n-1} = \frac{2\mathrm{i}}{z} + 2 - 2\,iz + 2\mathrm{z}^2 - \ldots + 2(-iz)^{n-1} + \ldots\ .$$

(a)(iii) $f(z) = \dfrac{2}{z(z-i)} = \dfrac{1}{\{(z-i)+i\}}\dfrac{2}{(z-i)} = \dfrac{2}{(z-i)^2}\dfrac{1}{1+\frac{i}{z-i}}$

$$= \frac{2}{(z-i)^2}\left\{\sum_{n=0}^{\infty}\left(\frac{-i}{z-i}\right)^n\right\}$$

since |i / (z-i)| < 1 on {z : |z - i| > 1} (Unit B3, Section 3, Para. 5)

Therefore the required Laurent series about i is

$$-2\sum_{n=0}^{\infty}\left(\frac{-i}{z-i}\right)^{n+2} = \frac{2}{(z-i)^2} - \frac{2\mathrm{i}}{(z-i)^3} - \frac{2}{(z-i)^4} - \ldots - 2\left(\frac{-i}{z-i}\right)^{n+2} - \ldots\ .$$

(b) 8 marks

(b)(i) The Laurent series for $g(z) = z^2 \sin(1/z)$ about 0 is

$$z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n-1}$$

Therefore the required series for g is $z - \frac{1}{6z} + \frac{1}{120z^3} - \cdots \quad z \in \mathbb{C} - \{0\}$

(b)(ii) g has an essential singularity at 0 since there are an infinite number of terms with negative powers of z. (Unit B4, Section 2, Para. 8)

(b)(iii) $z^2 \sin(1/z)$ is analytic on the punctured disc $\mathbb{C} - \{0\}$.

As C is a circle with centre 0 then (Unit B4, Section 4, Para. 2)

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz = 2\pi i\, a_{-1} = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}$$

where a_{-1} is the coefficient of z^{-1} in the Laurent series for g about 0.

(b)(iv)

$z^{2n} \sin(1/z)$ (n = 1, 2, 3, ..) is analytic on the punctured disc $\mathbb{C} - \{0\}$.

The Laurent series about 0 for $z^{2n} \sin(1/z)$ on this disc is

$$z^{2n} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left(\frac{1}{z}\right)^{2m+1} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left(\frac{1}{z}\right)^{2(m-n)+1} = \sum_{s-\infty}^{\infty} a_s z^s$$

As C is a circle with centre 0 then (Unit B4, Section 4, Para. 2)

$$\int_C z^{2n} \sin\left(\frac{1}{z}\right) dz = 2\pi i\, a_{-1} = 2\pi i\left(\frac{(-1)^n}{(2n+1)!}\right), \quad \text{for } n = 1, 2, 3, \ldots$$

Question 11

(a) 6 marks

Since $f(z)=\dfrac{\pi\cot\pi z}{9\left(z-\frac{2i}{3}\right)\left(z+\frac{2i}{3}\right)}$ then f has simple poles at z = ±2i/3.

By the cover-up rule (Unit C1, Section 1, Para. 3)

$$\text{Res}\left(f,\tfrac{2i}{3}\right)=\frac{\pi\cot(2i\pi/3)}{9\left(\frac{2i}{3}+\frac{2i}{3}\right)}=\frac{\pi\cot(2i\pi/3)}{12i}\text{ , and}$$

$$\text{Res}\left(f,\tfrac{-2i}{3}\right)=\frac{\pi\cot(-2i\pi/3)}{9\left(\frac{-2i}{3}-\frac{2i}{3}\right)}=\frac{\pi\cot(-2i\pi/3)}{-12i}\ .$$

Since sin(iz) = i sinh z and cos(iz) = cosh z then cot(iz) = - i coth(z).

Therefore $\text{Res}\left(f,\tfrac{2i}{3}\right)=-\dfrac{\pi\coth(2\pi/3)}{12}$ and

$$\text{Res}\left(f,\tfrac{-2i}{3}\right)=\frac{\pi\coth(-2\pi/3)}{12}=-\frac{\pi\coth(2\pi/3)}{12}\text{ . (Unit A2, Section 4, Para. 6)}$$

f(z) = g(z) / h(z) where $g(z)=\dfrac{\pi\cos\pi z}{9z^2+4}$ and h(z) = sin πz.

g and h are analytic at 0, h(0) = 0, and $h^{/}(0)=\pi\cos(0)=\pi\neq 0$.

Therefore by the g/h rule (Unit C1, Section 1, Para. 2)

$$\text{Res}(f,0)=\frac{g(0)}{h'(0)}=\frac{\pi*1}{4}*\frac{1}{\pi}=\frac{1}{4}\ .$$

[You could also use Unit C1, Section 4, Para 1 – last line]

(b) 8 marks

The method given in Unit C1, Section 4, Para. 1 will be used.

$f(z) = \pi \cot \pi z * \phi(z)$ where $\phi(z) = 1/(9z^2 + 4)$.

ϕ is an even function which is analytic on $\mathbb{C}$ except for simple poles at the non-integral points $z = \pm 2i/3$.

Let S_N be the square contour with vertices at $(N + ½)(\pm 1 \pm i)$.

On S_N we have $|z| \geq N + ½$ so, using the backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 2),

$$|9z^2 + 4| \geq |\,|9z^2| - 4\,| \geq 9(N + ½)^2 - 4 \geq 9N^2.$$

On S_N we also have $\cot \pi z \leq 2$ (Unit C1, Section 4, Para. 2) so on C_N

$$|f(z)| \leq \frac{\pi(2)}{9N^2} .$$

The length of the contour S_N is $4(2N + 1)$.

As f is continuous on the contour S_N then by the Estimation Theorem (Unit B1, Section 4, Para. 3) we have

$$\left|\int_{S_N} f(z)\,dz\right| \leq \frac{2\pi}{9N^2} 4(2N+1) = \frac{8\pi}{9N}\left(2 + \frac{1}{N}\right) .$$

Hence $\lim_{N \to \infty} \left|\int_{S_N} f(z)\,dz\right| = 0$.

Therefore the conditions specified in Unit C1, Section 4, Para. 1 hold so

$$\sum_{n=1}^{\infty} \frac{1}{9n^2+4} = -\frac{1}{2}\left(\operatorname{Res}(f, 0) + \operatorname{Res}(f, 2i/3) + \operatorname{Res}(f, -2i/3)\right)$$

$$= -\frac{1}{8} + \frac{\pi}{12} \coth \frac{2\pi}{3} .$$

(c) 4 marks

$$\sum_{n=-\infty}^{\infty} \frac{1}{9n^2+4} = \sum_{n=-\infty}^{-1} \frac{1}{9n^2+4} + \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{9n^2+4}$$

$$= \frac{1}{4} + 2\sum_{n=1}^{\infty} \frac{1}{9n^2+4} = \frac{\pi}{6} \coth \frac{2\pi}{3} .$$

Question 12

(a) 8 marks

(a)(i) The circle C has centre $\lambda = 0$, and radius $r = 2$. I shall take $\alpha = 1 + i$ as an inverse point with respect to the circle C and show that the corresponding inverse point $\beta = 2(1 + i)$.

Since C is not an extended line then $k \neq 1$. Therefore the equation $(\alpha-\lambda)\overline{(\beta-\lambda)}=r^2$ given in Unit D1, Section 3, Para. 7 holds.

Hence $(1+i)\overline{(\beta)}=4$. Taking the conjugate of both sides gives (1-i) $\beta = 4$ or $\beta = 2(1 + i)$..

Hence the given α and β are inverse point with respect to C.

(a)(ii)

$$g(\alpha)=\frac{2}{(1+i)-(1+i)}=\infty \text{ , and } g(\beta)=\frac{2}{2(1+i)-(1+i)}=1-i$$

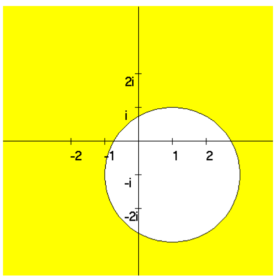
As α and β are inverse points with respect to the generalised circle C then $\hat{g}(\alpha)$ and { $\hat{g}(\beta)$¿ are inverse points with respect to $\hat{g}(C)$. (Unit D1, Section 3, Para. 6)

Therefore the centre of the circle $\hat{g}(C)$ is at $1 - i$ (Unit D1, Section 3, Para. 5). Since a point on C is mapped to a point on $\hat{g}(C)$ then $g(2)=\frac{2}{2-(1+i)}=\frac{2}{1-i}=1+i$ is on $\hat{g}(C)$. Therefore the radius of $\hat{g}(C)$ is $|(1 + i) - (1 - i)| = 2$.

The image of C under G is the boundary of the white circle in the diagram below.

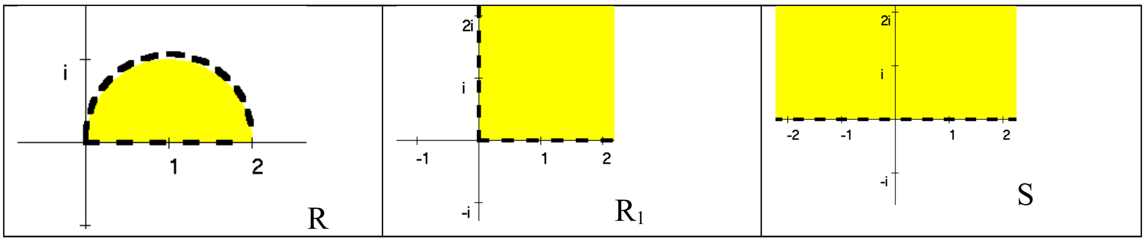
(a)(iii)

As $g(1 + i) = \infty$ then a point inside the circle C is mapped to a point outside $\hat{g}(C)$. Therefore D is the open shaded region with boundary $\hat{g}(C)$.



(b) 10 marks

(b)(i)



(b)(ii)

Using the formula for a transformation mapping points to the standard triple (Unit D1, Section 2, Para. 11) then the M★bius transformation $\hat{f}_1$ which maps 0, 1, and 2 to 0, 1, and ∞ respectively is

$$f_1(z) = \frac{(z-0)}{(z-2)}\frac{(1-2)}{(1-0)} = \frac{-z}{z-2}$$

Therefore the boundaries of R are mapped to extended lines in R_1. Since M★bius transformations are conformal these lines in R_1 meet at the origin at right-angles.

The line along the origin in R is mapped to the positive real-axis in R_1 since f(1) = 1. As we move from 0 to 1 in R the region to be mapped is on the left-hand side. As the transformation is conformal this must also be the case in R_1. Therefore is mapped to R_1 by f_1.

(b)(iii) $w = z_1^2$ is a conformal mapping from R_1 to S.

Therefore a conformal mapping from R to S is

$$f(z) = \left(\frac{-z}{z-2}\right)^2 .$$

(b)(iv) The point 0 belongs to the closure of R. Since at this point $f'(z) = 0$ then f is not conformal (Unit A4, Section 4, Para. 6).