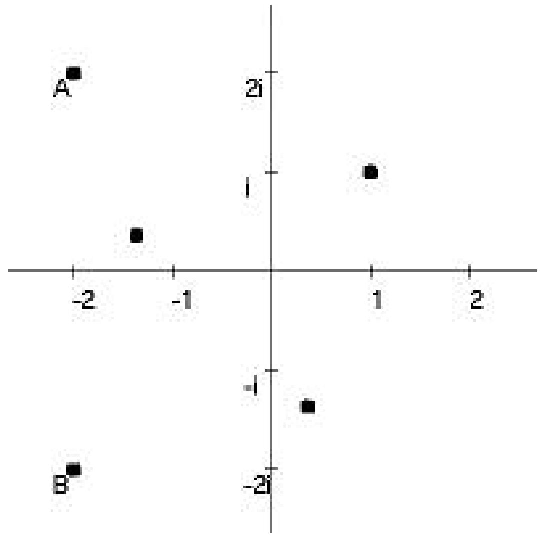
2001 Question 1

(a) 1 mark



The points A and B are α and its complex conjugate respectively. The unlabelled points are the cube roots.

(b) 2marks (Unit A1, Ex. 2.1(e))

$|\alpha| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ (Unit A1, Section 2, Para. 2)

Arg $\alpha = 3\pi/4$. (Unit A1, Section 2, Para. 8)

(c) 2 marks (Unit A1, Ex. 3.1(b)(ii))

The principal cube root of α is (Unit A1, Section 3, Para. 3)

$$\left(2\sqrt{2}\right)^{1/3}\left(\cos\left(\frac{1}{3}\frac{3\pi}{4}\right) + i\sin\left(\frac{1}{3}\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i$$

(d) 2 marks (Unit A1, Ex. 3.1(b)(ii))

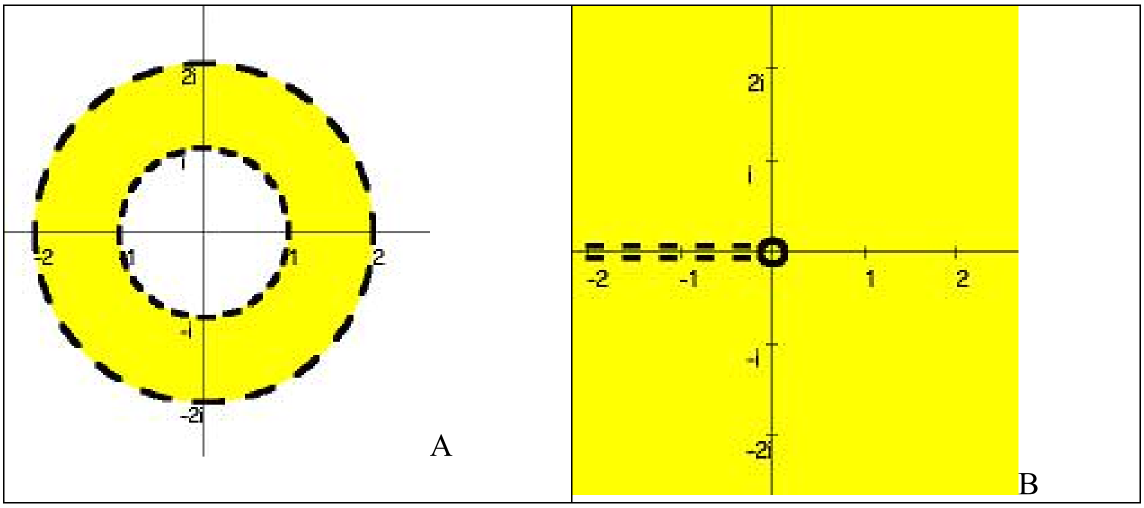
The other roots are found by rotating the principal root through $2\pi/3$ and $4\pi/3$. (Unit A1, Section 3, Para. 5). See diagram in part (a)

(e) 1 mark

$k = 4$

2001 Question 2

(a) 2 marks



(b) 6 marks

(b)(i) A and B are regions (Unit A3, Section 4, Paras. 6 and 7).

(b)(ii) f is analytic on A and B (Unit A4, Section 1, Para. 3, and Section 3 Para. 4).

[This is because f is analytic on $\mathbb{C} - \{0\}$ and neither A or B contain $\{0\}$. Remember that Arg z is not defined for z = 0 (Unit A1, Section 2, Para. 5).]

(b)(iii) B (Unit B1, Section 3, Para. 8 and Unit B2, Section 2, Para. 1).

[We can draw closed contours in A round the singularity at 0 so the integral is non-zero. 1/z is analytic on the simply-connected region B (Unit B2, Section 1, Para. 3) so by Cauchy's Theorem (Unit B2, Section 1, Para. 4) the integral is 0.]

(b)(iv) A (Unit A3, Section 3, Para. 3b).

[On B we can get as close to 0 as we want e.g. the sequence $\{z_n = 1/n\}$.
Therefore |f| is unbounded]

2001 Question 3

(a) 3 marks

(a)(i) The standard parametrization for Γ_1 (Unit A2, Section 2, Para. 3) is
$\gamma_1(t) = (1 - t)(-1) + ti = (t - 1) + ti, \quad t \in [0, 1]$

(a)(ii) $\gamma_1'(t) = 1 + i$.

As γ is differentiable on [0, 1], γ' is continuous on [0, 1], and γ' is non-zero on [0, 1] then γ is a smooth path (Unit A4, Section 4, Para. 3).

As γ is a smooth parametrization then (Unit B1, Section 2, Para. 1)

$$\int_{\Gamma_1} \operatorname{Re} z \, dz = \int_0^1 \{\operatorname{Re} \gamma_1(t)\} \gamma_1'(t) dt$$

$$= \int_0^1 (t-1)(1+i)\, dt$$

$$= (1+i)\left[\frac{(t-1)^2}{2}\right]_0^1 = -\frac{1+i}{2}$$

(b) 5 marks

The length of the contour Γ_2, $L = |(1 + i) - (1 - i)| = |2i| = 2$.

Using the Triangle Inequality (Unit A2, Section 5, Para. 3a) then for z on the contour Γ_2

$$|\operatorname{Log} z| = |\log_e|z| + i \operatorname{Arg} z| \qquad \text{(Unit A2, Section 5, Para. 1)}$$

$$\leq |\log_e|z|\,| + |\operatorname{Arg} z|$$

$$\leq \log_e \sqrt{2} + \frac{\pi}{4} < 3 \qquad \text{(Less writing)}$$

Using the Backwards form of the Triangle Inequality (Unit A2, Section 5, Para. 3c) then for $z \in \Gamma_2$

$$|5 + z^2| \geq \left|5 - |z|^2\right| \geq |5 - 2| = 3 \text{ since } 1 \leq |z| \leq 2^{1/2}.$$

Putting $f(z) = \dfrac{\operatorname{Log} z}{5 + z^2}$ then on Γ_2 we have $|f(z)| \leq \dfrac{3}{3} = 1 = M$.

By the Quotient Rule (Unit A3, Section 2, Para. 5) f(z) is continuous on $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and hence on the contour Γ_2. Therefore by the Estimation Theorem (Unit B1, Section 4, Para. 3)

$$\left|\int_{\Gamma_2} \frac{\operatorname{Log} z}{5 + z^2} dz\right| \leq ML = 1 * 2 = 2.$$

2001 Question 4

(a) 4 marks

(i) The Taylor series for sinh and exp (Unit B3, Section 3, Para. 5) are

$$\sinh z = z + \frac{z^3}{3!} + \ldots \qquad \text{for } z \in \mathbb{C}.$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \qquad \text{for } z \in \mathbb{C}.$$

By the Composition theorem for Taylor Series (Unit B3, Section 4, Para. 3)

$$e^{\sinh z} = 1 + \left(z + \frac{z^3}{3!} + \ldots\right) + \frac{1}{2!}\left(z + \frac{z^3}{3!} + \ldots\right)^2 + \frac{1}{3!}\left(z + \frac{z^3}{3!} + \ldots\right)^3 + \ldots$$

$$= 1 + z + \frac{z^2}{2} + z^3\left(\frac{1}{3!} + \frac{1}{3!}\right) + \ldots$$

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$

(ii) Since sinh z and e^z are both entire functions then by the Composition rule so is $e^{\sinh z}$ (Unit A4, Section 3, Para. 1). Therefore the Taylor series for f is also valid for $z \in \mathbb{C}$. (Unit B3, Section 3, Para. 3)

(b) 4 marks

$$g(z) = \frac{1}{z^2}\left\{\frac{1}{1 + 1/z^2}\right\}$$

$$= \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{-1}{z^2}\right)^n \qquad \text{since } |z| > 1.$$

$$= \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \ldots + (-1)^n\left(\frac{1}{z}\right)^{2n+2} + \ldots$$

2001 Question 5

(a) 4 marks

Since $z^3 - 1$ has zeros at $z = 1$, $e^{2\pi i/3}$, and $e^{4\pi i/3}$ then f also has simple poles at these points.

Let $g(z) = 1$ and $h(z) = z^3 - 1$. Then $h'(z) = 3z^2$.

If α is one of the poles then g and h are analytic at α, $h(\alpha) = 0$, and $h'(\alpha) = 3\alpha^2 \neq 0$. Therefore by the g/h rule (Unit C1, Section 1, Para. 2)

$$\text{Res}(f,1) = \frac{1}{3}$$

$$\text{Res}\left(f, e^{2\pi i/3}\right) = \frac{1}{3e^{4\pi i/3}} = \frac{1}{3}e^{2\pi i/3}$$

$$\text{Res}\left(f, e^{4\pi i/3}\right) = \frac{1}{3e^{8\pi i/3}} = \frac{1}{3}e^{-2\pi i/3}$$

(b) 4 marks

I shall use the result given in Unit C1, Section 3, Para. 8.

Let $p(t) = 1$, $q(t) = t^3 - 1$, and $f(t) = \dfrac{p(t)}{q(t)}$.

p and q are polynomial functions such that the degree of q exceeds that of p by at least 2, and the pole of p/q on the real axis is simple. Therefore

$$\int_{-\infty}^{\infty} \frac{1}{t^3 - 1}\,dt = 2\pi i S + \pi i T$$

where S is the sum of the residues of f at the poles in the upper half-plane, and
T is the sum of the residues of f at the poles on the real axis.

As $S = \text{Res}(f, e^{i2\pi/3})$ and $T = \text{Res}(f, 1)$.

$$\int_{-\infty}^{\infty} \frac{1}{t^3 - 1}\,dt = 2\pi i\left(\frac{e^{2\pi i/3}}{3}\right) + \pi i\left(\frac{1}{3}\right) \qquad \text{using part (a).}$$

$$= \frac{2\pi i}{3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \frac{\pi i}{3} = -\frac{\sqrt{3}\pi}{3}$$

2001 Question 6

(a) 2 marks

Using the Triangle Inequality (Unit A1 Section 5, Para 3) we have

$$|\sinh z| = \left|\frac{e^z - e^{-z}}{2}\right| \le \frac{1}{2}\left\{|e^z| + |e^{-z}|\right\}$$

$$= \frac{1}{2}\left\{e^{\mathrm{Re}\,z} + e^{\mathrm{Re}(-z)}\right\} \quad \text{(Unit A2, Section 4, Para. 2)}$$

$$\le \frac{1}{2}\left\{e^{|\mathrm{Re}\,z|} + e^{|\mathrm{Re}(-z)|}\right\} = e^{|\mathrm{Re}\,z|} \quad \text{as } |\mathrm{Re}\,z| = |\mathrm{Re}\,(-z)|$$

(b) 4 marks

I shall use Weierstrass' M-test (Unit C3, Section 3, Para. 5) with $\phi_n(z) = \frac{\sinh z}{n^2+1}$ where n is an integer.

On E, $|\phi_n(z)| = \left|\frac{\sinh z}{n^2+1}\right| \le \frac{e^{|\mathrm{Re}\,z|}}{n^2+1}$ using part (a)

$$\le \frac{e^3}{n^2+1} \quad \text{as } |\mathrm{Re}\,z| \le 3 \text{ on E.}$$

$$\le \frac{e^3}{n^2}$$

Therefore the 1st assumption of Weierstrass' M test holds if we set $M_n = \frac{e^3}{n^2}$.

Since $\sum_{n=1}^{\infty} M_n = e^3 \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (Unit B3, Section 1, Para. 8) then $\sum_{n=1}^{\infty} M_n$ is convergent. Therefore the 2nd assumption of the M test also holds.

Hence by the M-test $\sum_{n=1}^{\infty} \frac{\sinh z}{n^2+1}$ converges uniformly on E.

(c) 2 marks

Since the functional equation of the Gamma function (Unit C3, Section 4, Para. 2) holds on $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$ (Unit C3, Section 4, Para. 3) so

$$\tfrac{1}{2}\sqrt{\pi} = \Gamma\left(\tfrac{3}{2}\right) \quad \text{(Unit C3, Section 4, Para. 4)}$$

$$= \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\Gamma\left(-\tfrac{1}{2}\right)$$

So $\Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}$.

2001 Question 7

(a) 1 mark

q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C}$ and the conjugate velocity function $\overline{q}(z) = z + i$ is analytic on $\mathbb{C}$. Therefore q is a model flow on $\mathbb{C}$ (Unit D2, Section 1, Para. 14).

(b) 4 marks

The complex potential function Ω is a primitive of $\overline{q}(z)$ (Unit D2, Section 2, Para. 1). Therefore the complex potential function $\Omega(z) = \frac{z^2}{2} + iz$ and the stream function

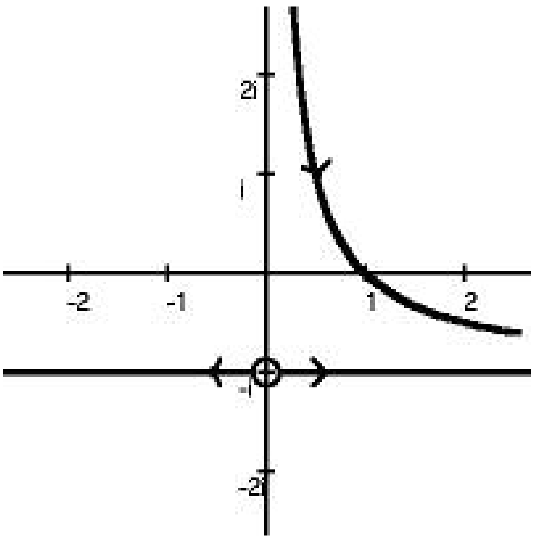
$$\Psi(x, y) = \operatorname{Im}\Omega(z) = xy + x \quad \text{(Unit D2, Section 4, Para. 4)}$$

A streamline through 1 is given by $x(y+1) = \Psi(1,0) = 1$.
So $y = (1/x) - 1$
The velocity function at 1 is q(1) = 1 – i (south-east)

A streamline through -1 - i is given by $x(y+1) = \Psi(-1,-1) = 0$.
Since the streamline passes through –1 – i then it must be y = -1.
The velocity function at –1 - i is q(-1 - i) = -1 (Left)

(c) 3 marks

A degenerate streamline (Unit D2, Section 1, Para. 4) has q(z) = 0 at some point on the streamline. This occurs when z = -i.



The circle indicates the stagnation point.

2001 Question 8

(a) 3 marks

$f^1(i) = i^3 + i = 0.$
$f^2(i) = i.$

Since $f^2(i) = i$ then i is a periodic point with period 2 (Unit D3, Section 2, Para. 7).

$(f^2)'(i) = f'(i) * f'(f^1(i)) = f'(i) * f'(0)$ (Unit D3, Section 2, Para. 8)

$f'(z) = 3z^2$ so $f'(0) = 0$.

Therefore since $|(f^2)'(i)| = 0$ then i is a super-attracting point (Unit D3, Section 2, Para. 10).

(b) 5 marks

(b)(i) Same as 2002 Question 8(b)(ii).

(b)(ii)

$P_c(0) = -1 - i.$
$P_c^2(0) = (-1-i)^2 + (-1-i) = 2i + (-1-i) = -1+i.$
$P_c^3(0) = (-1+i)^2 + (-1-i) = -2i + (-1-i) = -1-3i.$

As $\left|P_c^3(0)\right| > 2$ then c does not lie in the Mandelbrot set (Unit D3, Section 4, Para. 5).

2001 Question 9

(a) 8 marks

Putting $z = x + iy$ we have

$f(z) = (x + iy)^2 + 3y^2 + i6x^2$
$= (x^2 - y^2 + 3y^2) + i(2xy + 6x^2)$
$= u(x, y) + i\, v(x, y)$
where $u(x,y) = x^2 + 2y^2$, and $v(x,y) = 2xy + 6x^2$.

$$\frac{\partial u}{\partial x}(x,y) = 2x\,,\ \frac{\partial u}{\partial y}(x,y) = 4y\,,\ \frac{\partial v}{\partial x}(x,y) = 2y + 12x\,,\ \frac{\partial v}{\partial y}(x,y) = 2x$$

f is not differentiable at a point unless the Cauchy-Riemann equations (Unit A4, Section 2, Para. 1) hold.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ always holds.

$\frac{\partial v}{\partial x}(\alpha,\beta) = 2\beta + 12\alpha = -4\beta = -\frac{\partial u}{\partial y}(\alpha,\beta)$ holds when $\beta = -2\alpha$

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

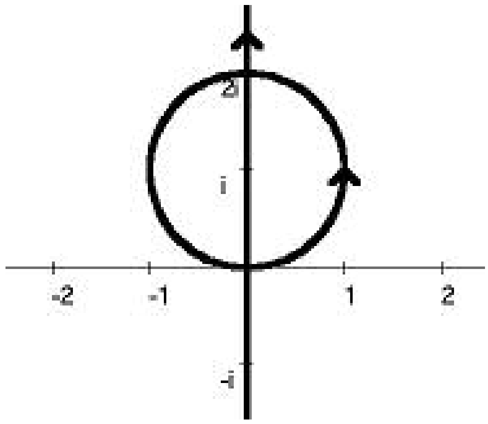
1. exist on $\mathbb{C}$
2. are continuous on the line $y = -2x$.
3. satisfy the Cauchy-Riemann equations on this line

then, by the Cauchy-Riemann Converse Theorem (Unit A4, Section 2, Para. 3), f is differentiable on the line $\{ x - 2ix : x \in \mathbb{R}\}$

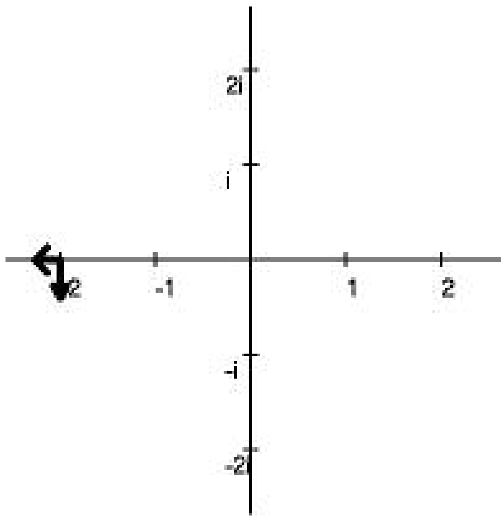
(b) 10 marks

(i) Since g is a polynomial then g is entire (Unit A4, Section 1, Para. 7) and $g'(z) = 2z$ on $\mathbb{C}$. As $g(z) \neq 0$ when $z \neq 0$ then g is conformal on $\mathbb{C} - \{0\}$ (Unit A4, Section 4, Para. 6).

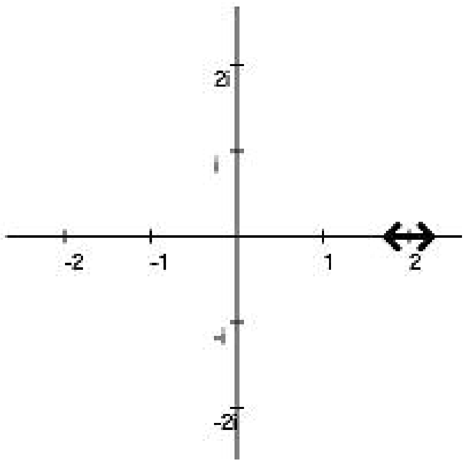
(ii) As $\pi/2$ is in the domain of γ_1 then $\gamma_1(\pi/2) = i + \exp(i\pi/2) = 2i$.
As 2 is in the domain of γ_2 then $\gamma_2(2) = 2i$.
Therefore Γ_1 and Γ_2 meet at the point $2i$.



(iii) As g is analytic on $\mathbb{C}$ and $g^{/}(2i) \neq 0$ then a small disc centred at 2i is mapped approximately (Unit A4, Section 1, Para. 11) to a small disc centred at $g(2i) = -4 + 2 = -2$. The disc is rotated by $\text{Arg}\,(g^{/}(2i)) = \text{Arg}\,4i = \pi/2$, and scaled by a factor $|\,g^{/}(2i)| = 4$.



(iv) $g(0) = 2$.



$g(\Gamma_1)$ is the arrow to the right.
[As $g^{/}(0) = 0$ the mapping is not conformal.]

2001 Question 10

(a) 2 marks

Since exp (z) is an entire function then f is defined when 1/(z+1) is defined. Therefore f only has one singularity and this is at z = -1.

Therefore the Laurent series for f is $\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z+1}\right)^n$ for $|z + 1| > 0$.

As there are an infinite number of terms with a negative power of (z + 1) then f has an essential singularity at –1 (Unit B4, Section 2, Para. 8).

(b) 16 marks

(b)(i) Let R = {z : |z| < 1 }.

As R is a simply-connected region, f is analytic on R, and C_1 is a closed contour in R then by Cauchy's Theorem (Unit B2, Section 1, Para. 4)

$$\int_{C_1} f(z)\,dz = 0$$

(b)(ii) $\displaystyle\int_{C_1} \frac{f(z)}{(4z+1)^2}\,dz = \frac{1}{16}\int_{C_1} \frac{f(z)}{(z+\frac{1}{4})^2}\,dz$

Let R = {z : |z| < 1 }.

R is a simply-connected region, f is analytic on R, and C_1 is a closed contour in R.
As –1/4 lies inside C_1 then by Cauchy's n^{th} derivative formula (Unit B2, Section 3, Para. 1) with n = 1 and α = -1/4 we have

$$\int_{C_1} \frac{f(z)}{(4z+1)^2}\,dz = \frac{1}{16}\left\{2\pi i f'\left(-\tfrac{1}{4}\right)\right\}$$

Since $f'(z) = -\frac{1}{(z+1)^2}\exp\left(\frac{1}{z+1}\right)$ then

$$\int_{C_1} \frac{f(z)}{(4z+1)^2}\,dz = -\frac{1}{16}\left\{2\pi i \frac{16}{9}\exp\left(\tfrac{4}{3}\right)\right\} = -\frac{2\pi i}{9}\exp\left(\tfrac{4}{3}\right)$$

(b)(iii)

Let **R** = $\mathbb{C}$.

R is a simply-connected region and f is analytic on **R** except for a finite number of singularities. C_2 is a simple-closed contour in **R** not passing through any singularities. Therefore by Cauchy's Residue Theorem (Unit C1, Section 2, Para. 1)

$$\int_{C_2} f(z)\,dz = 2\pi i \,\text{Res}(f,-1) = 2\pi i(1)$$

since, from part (a), the coefficient of $(z+1)^{-1}$ in the Laurent series for f is 1.

(b)(iv)

$g(z) = \dfrac{f(z)}{z}$ has a simple pole at z = 0 and an essential singularity at z = -1.

Since $\mathbb{C}$ is a simply-connected region and g is analytic on $\mathbb{C}$ except at z = 0 and z = -1, C_2 is a simple-closed contour in $\mathbb{C}$ not passing through either of these singularities. Then by Cauchy's Residue Theorem (Unit C1, Section 2, Para. 1)

$$\int_{C_2} g(z)\,dz = 2\pi i(\text{Res}(g,0) + \text{Res}(g,-1))$$

$$\text{Res}(g,0) = \lim_{z \to 0} (z-0)g(z) = f(0) = e\,.$$

$$\frac{f(z)}{z} = -\frac{f(z)}{1-(z+1)} = -\left\{\sum_{n=0}^{\infty}(z+1)^n\right\}\left\{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z+1}\right)^n\right\} \text{ when } |\, z+1 \,| < 1.$$

The coefficient of $(z + 1)^{-1}$ in this equation is

$$-\frac{1}{1!}-\frac{1}{2!}-\frac{1}{3!}-\ldots = -e^1 + 1\,.$$

Therefore Res (g, -1) = -e + 1.

Hence $\int_{C_2} g(z)\,dz = 2\pi i\,(e - e + 1) = 2\pi i\,.$

2001 Question 11

(a) 7 marks

(a)(i)

Let R = $\mathbb{C}$.
Since f is analytic on the region R (Unit A4, Section 1, Para. 7) and $0 \in R$ and the Taylor series of f about 0 is

$$f(z) = -z + z^3$$

then, by the Local Mapping Theorem (Unit C2, Section 3, Para. 4), f is one-one near 0.

(a)(ii)

Since $f'(0) = 1 \neq 0$ then using the strategy for inverting a Taylor series (Unit C2, Section 3, Para. 8) we have

$$z = b_1(-z + z^3) + b_2(-z + z^3)^2 + b_3(-z + z^3)^3 + b_4(-z + z^3)^4 + b_5(-z + z^3)^5 + \ldots$$

where the b_i are the coefficients of the Taylor series for f^{-1} about f(0) = 0.

As f is odd then so is f^{-1} so $b_{2n} = 0$ for all $n \in \mathbb{Z}$.

Equating powers of z we have

z: $1 = -b_1 \Rightarrow b_1 = -1$.
z^3: $0 = b_1 - b_3 \Rightarrow b_3 = b_1 = -1$.
z^5: $0 = 3b_3 - b_5 \Rightarrow b_5 = -3$.

Therefore $f^{-1}(z) = -z - z^3 - 3z^5 - \ldots$.

(b) 7 marks

Ler R = {z: |z| < 1}.

Since g is defined on the bounded region R and continuous on $\overline{R}$ then, by the Maximum Principle (Unit C2, Section 4, Para. 4), there exists an $\alpha \in \partial R$ such that

$$|g(z)| \leq |g(\alpha)| \quad \text{for } z \in \overline{R}.$$

$\max\{|g(z)| : |z| \leq 1\}$
$= \max\{|g(z)| : z = e^{it}, t \in [0, 2\pi]\}$ (on ∂R)
$= \max\{|e^{2it} + i| : t \in [0, 2\pi]\}$
$= \max\{|\cos(2t) + i(1 + \sin 2t)| : t \in [0, 2\pi]\}$
$= \max\{(\cos^2(2t) + 1 + \sin^2(2t) + 2 \sin 2t)^{1/2} : t \in [0, 2\pi]\}$
$= \max\{(2 + 2 \sin 2t)^{1/2} : t \in [0, 2\pi]\}$
$= 2$ when $t = \pi/4$, or $t = 5\pi/4$.

[[OR $|e^{2it} + i|] \leq |e^{2it}| + |i| = 1 + 1 = 2$. Since $|e^{2it} + i| = 2$ when $t = \pi/4$, or $t = 5\pi/4$ then

max {|g(z)| : |z| ≤ 1} = 2]

By the Triangle Inequality (Unit A1, Section 5, Para. 2)

$$|g(z)| = |z^2 + i| \le |z^2| + |i| < 2 \text{ if } |z| < 1.$$

Therefore g(z) only occurs its maximum value of 2 on the boundary of |z| ≤ 1 and this occurs at

$$z = \pm\frac{1}{\sqrt{2}}(1+i).$$

(c) 4 marks

(c)(i)

True.

As h is one-one on D then it is not constant on the region D. As h is analytic and non-constant on D then by the Corollary to the Open Mapping Theorem (Unit C2, Section 3, Para. 2) h(D) is also a region.

(c)(ii)

False.

$h(z) = \frac{1}{1-z}$ is analytic on D.

If $z_1, z_2 \in D$ then

$$h(z_1) = h(z_2) \Rightarrow \frac{1}{1-z_1} = \frac{1}{1-z_2} \Rightarrow 1 - z_2 = 1 - z_1 \Rightarrow z_1 = z_2.$$

Therefore h is one-one on D.

Assume | h(z) | is bounded above by M > 0. Since h(1-1/2M) = 2M > M then the assumption that h is bounded is incorrect.

Therefore h(D) is not bounded.

2001 Question 12

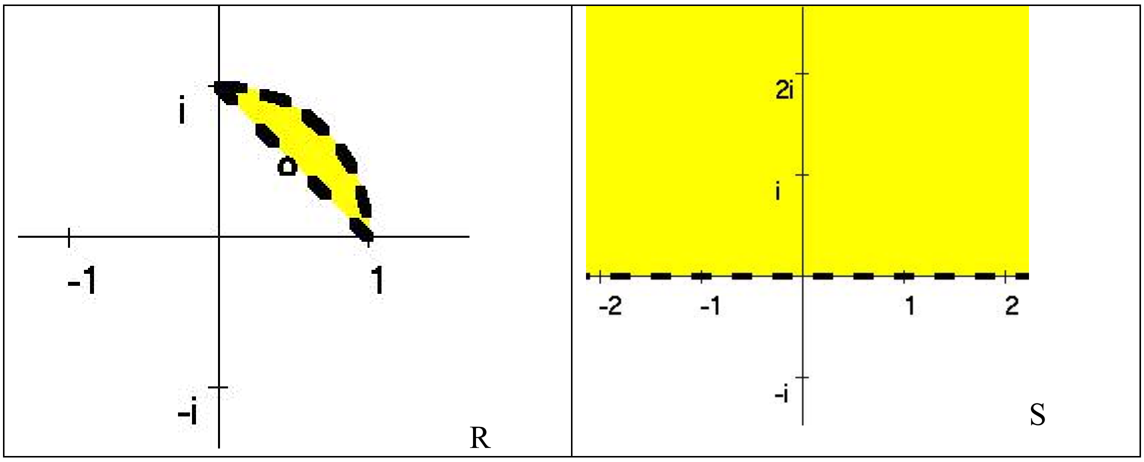
(a) 3 marks

Using the formula for a transformation mapping points to the standard triple (Unit D1, Section 2, Para. 11) then the Möbius transformation $\hat{f}_1$ which maps i, ½(i + 1) , and 1 to 0, 1, and ∞ respectively is

$$f_1(z) = \frac{(z-i)}{(z-1)}\frac{\left(\frac{1}{2}(1+i)-1\right)}{\left(\frac{1}{2}(1+i)-i\right)} = \frac{(z-i)}{(z-1)}\frac{\frac{1}{2}(-1+i)}{\frac{1}{2}(1-i)} = \frac{-z+i}{z-1}$$

(a) 15 marks

(b)(i)



(b)(ii) Since $\hat{f}_1$ maps i to 0 and 1 to ∞ then the straight and curved boundaries of R are mapped to extended lines originating at the origin.

From part (a), ½(1 +i) is mapped to a point on the positive real-axis then the straight boundary is mapped to the non-negative x-axis.

At z = i the angle between the boundary lines of R are at an angle of π/4. Therefore as the transformation is conformal then this is also the angle at the origin of the transformed lines. Going along the straight line boundary in R from i towards 1 the region to be mapped is on the left. Therefore the image of the region is above the non-negative real axis.

Therefore the image of R under $\hat{f}_1$ is $R_1 = \{z \in \mathbb{C}: 0 < \text{Arg } z < \pi/4\}$

(b)(iii) A conformal mapping from R_1 onto S is the power function $w = g(z) = z_1^4$. Since the combination of conformal mapping is also conformal then a conformal mapping from R to S is

$$f(z) = \left(\frac{-z + i}{z - 1} \right)^4$$

(b)(iv)

Since $f^{-1} = (g \circ f_1)^{-1} = (f_1^{-1} \circ g^{-1})$ then using Unit D1, Section 2, Para. 6 we have

$$f^{-1}(z) = \frac{z^{1/4} + i}{z^{1/4} + 1}$$