2000 Question 1

(a) 2 marks

(a)(i) $|\alpha| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$ (Unit A1, Section 2, Para. 2)

(a)(ii) $\text{Arg}\,\alpha = -3\pi/4$. (Unit A1, Section 2, Para. 8)

(b) 6 marks

(b)(i) $\alpha = 2\sqrt{2}\exp(-3i\pi/4)$

$$\frac{1}{\alpha} = \frac{1}{2\sqrt{2}}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) = -\frac{1}{4} + i\frac{1}{4}$$ (Unit A1, Section 2, Para. 12)

(b)(ii) The principal value of $\alpha^{1/3}$ is (Unit A1, Section 3, Para 3)

$$\left(2\sqrt{2}\right)^{1/3}\left(\cos\left(\frac{1}{3}\left(-\frac{3\pi}{4}\right)\right) + i\sin\left(\frac{1}{3}\left(-\frac{3\pi}{4}\right)\right)\right)$$

$$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 1 - i$$

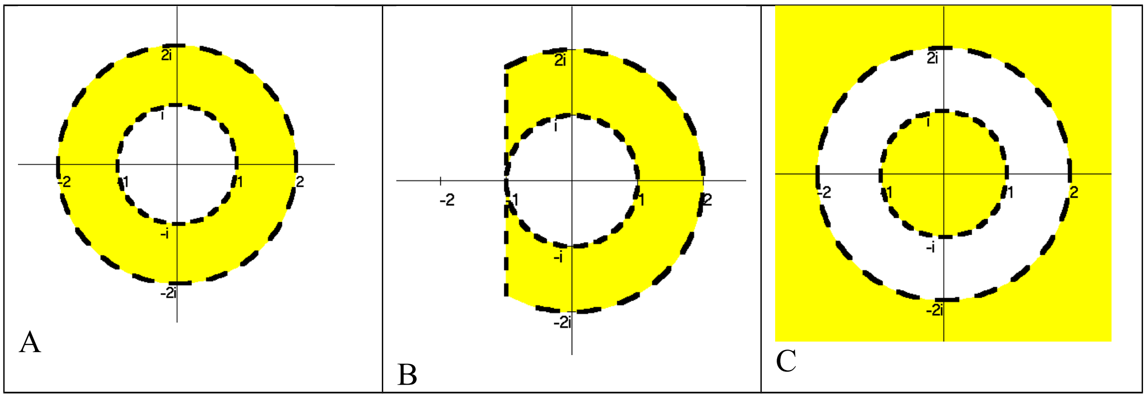
(b)(iii) $\text{Log}\,\alpha = \log_e\left(2\sqrt{2}\right) + i(-3\pi/4) = \frac{3}{2}\log_e 2 - \frac{3\pi}{4}i$
(Unit A2, Section 5, Para. 1)

(b)(iv) $\text{Log}(\alpha^3) = \text{Log}\left(16\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\right)$ as $-\frac{9}{4}\pi = -\frac{1}{4}\pi - 2\pi$

$$= \log_e\left(16\sqrt{2}\right) - \frac{\pi}{4}i = \frac{9}{2}\log_e 2 - \frac{\pi}{4}i$$ (Unit A2, Section 5, Para. 1)

2000 Question 2

(a) 3 marks



(b) 3 marks

(b)(i) A, B, and C are open (Unit A3, Section 4, Para. 1).

(b)(ii) A and B are regions (Unit A3, Section 4, Paras. 6 and 7).

(b)(iii) B is a simply-connected region (Unit B2, Section 1, Para. 3).

(c) 2marks

D = {0, 1}

[Since $\mathbb{C}$ – D is a region (Unit A3, Section 4, Paras. 7 and 8) then it is open. Therefore D is closed (Unit A3, Section 5, Para. 1)]

2000 Question 3

(a) 3 marks

A parametrization for the circle C is (Unit A2, Section 2, Para. 3)

$$\gamma(t) = 2e^{it} \qquad (t \in [0, 2\pi])$$

$\gamma'(t) = 2ie^{it}$

As γ is differentiable on $[0, 2\pi]$, γ' is continuous on $[0, 2\pi]$, and γ' is non-zero on $[0, 2\pi]$ then γ is a smooth path (Unit A4, Section 4, Para. 3).

Since γ is a smooth path then (Unit B1, Section 2, Para. 1)

$$\int_C \bar{z}\,dz = \int_0^{2\pi} \overline{\gamma(t)}\,\gamma'(t)\,dt = \int_0^{2\pi} 2e^{-it}\left(2ie^{it}\right)dt = 4i\int_0^{2\pi} dt = 8\pi i$$

(b) 5 marks

The length of the circle C, $L = 2\pi * 2 = 4\pi$.

Using the Triangle Inequality (Unit A1, Section 5, Para. 3b) then for z on the contour C

$$|\sin z| = \left|\frac{e^{iz} - e^{-iz}}{2i}\right| \le \frac{1}{2}\left\{\left|e^{iz}\right| + \left|e^{-iz}\right|\right\} \qquad \text{(Unit A2, Section 4, Para. 4)}$$

$$= \frac{1}{2}\left\{\left|\exp(\operatorname{Re}(iz))\right| + \left|\exp(\operatorname{Re}(-iz))\right|\right\} \qquad \text{(Unit A2, Section 4, Para. 2)}$$

$$= \frac{1}{2}\left\{\left|e^{-y}\right| + \left|e^{y}\right|\right\} \qquad \text{where } z = x + iy$$

$$\le \frac{1}{2}\left\{e^2 + e^2\right\} = e^2.$$

Using the Backwards form of the Triangle Inequality (Unit A1, Unit 5, Para. 3c) then for $z \in C$

$$\left|1 + z^6\right| \ge \left|1 - |z|^6\right| = |1 - 64| = 63$$

Putting $f(z) = \dfrac{\sin z}{1 + z^6}$ then on the circle C we have $|f(z)| \le \dfrac{e^2}{63} = M$.

By the Quotient Rule (Unit A3, Section 2, Para. 5) f(z) is continuous on $\mathbb{C} - \{z : |z| = 1\}$ and hence on the circle C. Therefore by the Estimation Theorem (Unit B1, Section 4, Para. 3)

$$\left|\int_\Gamma \frac{\sin z}{1 + z^6}dz\right| \le ML = \frac{e^2}{63} * 4\pi = \frac{4\pi e^2}{63}$$

2000 Question 4

(a) 3 marks

f is analytic on $\mathbb{C}$ - {-i}.

R = {z : | z | < 1} is a simply-connected region, f is an analytic on **R**, and C is a simple-closed contour in **R**.

Therefore by Cauchy's Theorem (Unit B2, Section 1, Para. 4) we have

$$\int_C f(z)\,dz = 0$$

(b) 5 marks

Let $g(z) = z\exp(z^2)$. g is a function which is analytic on the simply-connected region $\mathbb{C}$ (Unit B2, Section 1, Para. 3).

The contour C is a simple-closed contour in $\mathbb{C}$. Since the zero of z + i is inside the circle C then using Cauchy's n[th] Derivative Formula (Unit B2, Section 3, Para. 1), with n = 2 and α = -i we have

$$\int_C \frac{z\exp(z^2)}{(z+i)^3}\,dz = \int_C \frac{g(z)}{(z+i)^3}\,dz = \frac{2\pi i}{2!}g^{(2)}(i)$$

$$g'(z) = \exp(z^2) + 2z^2\exp(z^2) = (1+2z^2)\exp(z^2)$$

$$g''(z) = 4z\exp(z^2) + 2z(1+2z^2)\exp(z^2) = (6z+4z^3)\exp(z^2)$$

So $g''(-i) = (-6i+4i)\exp(-1) = -2ie^{-1}$

Hence $\displaystyle\int_C \frac{z\exp(z^2)}{(z+i)^3}\,dz = \frac{2\pi i}{2!} * (-2ie^{-1}) = \frac{2\pi}{e}$

2000 Question 5

(a) 4 marks

f is an analytic function which has simple poles at $\pm 3i$.

$$\operatorname{Res}(f,3i) = \lim_{z \to 3i} (z-3i)f(z) = \frac{e^{2i(3i)}}{(3i+3i)} = -i\frac{e^{-6}}{6}$$ Unit C1, Section 1, Para. 1

$$\operatorname{Res}(f,-3i) = \lim_{z \to -3i} (z+3i)f(z) = \frac{e^{2i(-3i)}}{(-3i-3i)} = i\frac{e^{6}}{6}$$

[or use the cover-up rule (Unit C1, Section 1, Para. 3)]

(b) 4 marks [Unit C1, Problem 3.12.]

I shall use the result given in Unit C1, Section 3, Para. 9.

Let $p(t) = 1$, $q(t) = t^2 + 9$, $f(t) = \frac{p(t)}{q(t)}\exp(ikt)$ where $k = 2$.

Since p and q are polynomials, the degree of q exceeds that of p by at least 1, there are no poles on the real axis and $k > 0$ then

$$\int_{-\infty}^{\infty} \frac{1}{t^2+9} e^{i2t} dt = 2\pi i S + \pi i T$$

where S is the sum of the residues of f at the poles in the upper half-plane, and
T is the sum of the residues of f at the poles on the real axis.

As S = Res(f, 3i) and T = 0 then

$$\int_{-\infty}^{\infty} \frac{1}{t^2+9} e^{i2t} dt = 2\pi i\left(-\frac{e^{-6}}{6} i\right) = \frac{\pi}{3} e^{-6}$$

So taking the real part of the last equation we have

$$\int_{-\infty}^{\infty} \frac{\cos 2t}{t^2+9} dt = \operatorname{Re}\left\{\int_{-\infty}^{\infty} \frac{e^{i2t}}{t^2+9} dt\right\} = \frac{\pi}{3} e^{-6}.$$

2000 Question 6

(a) 7 marks

(a)(i)

Let $g_1(z) = 7$.
For z on the contour C_1 then, using the Triangle Inequality (Unit A1 Section 5, Para 3),

$$| f(z) - g_1(z) | = |z^5 + 5iz^3 | \le |z^5| + |5iz^3| = 1 + 5$$
$$< 7 = | g_1(z) |.$$

Since f is a polynomial then it is analytic on the simply-connected region $\boldsymbol{R} = \mathbb{C}$.
Also as C_1 is a simple-closed contour in $\boldsymbol{R}$ then by Rouche's theorem (Unit C2, Section 2, Para. 4) f has the same number of zeros as g_1 inside the contour C_1. Therefore f has no zeros inside C_1.

(a)(ii)

Let $g_2(z) = 5iz^3$.
For z on the contour C_2 then, using the Triangle Inequality,

$$| f(z) - g_2(z) | = |z^5 + 7| \le |z^5| + 7 = 32 + 7 < 40 = | g_2(z) |.$$

As C_2 is a simple-closed contour in $\boldsymbol{R}$ then by Rouche's theorem f has the same number of zeros as g_2 inside the contour C_2. Therefore f has 3 zeros inside C_2.

(b) 1 mark

When $|z| = 3$ then, using the Triangle Inequality,

$$243 = |z^5| > 135 + 5 = |5iz^3| + 7 \ge |5iz^3 + 7|.$$

Therefore repeating a similar argument to those in part (a) with $g_3(z) = 5iz^3 + 7$ we can show all the zeros lie inside the circle $\{z: |z| = 3\}$.

Therefore M = 3 is a suitable answer.

2000 Question 7

(a) 1 mark

q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C} - \{0\}$ and the conjugate velocity $\overline{q}(z) = 3/z$ is analytic on $\mathbb{C} - \{0\}$. Therefore (Unit D2 Section 1, Para. 14) q is a model fluid flow on $\mathbb{C} - \{0\}$.

(b) 5 marks

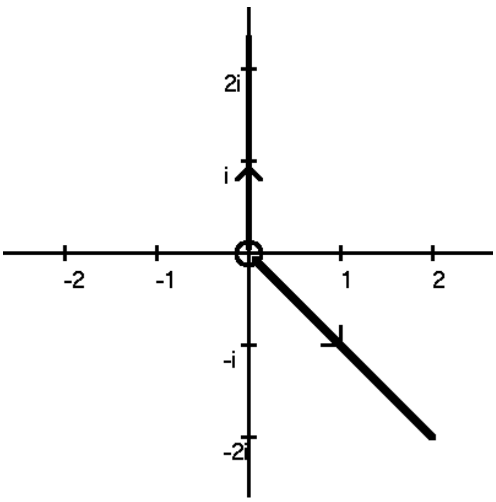
The complex potential function Ω is a primitive of $\overline{q}(z)$ (Unit D2, Section 2, Para. 1). Therefore the complex potential function $\Omega(z) = 3\,\mathrm{Log}\,z$ and the stream function

$\Psi(x,y) = \mathrm{Im}\,\Omega(z)$ (Unit D2, Section 4, Para. 4)

$= 3\,\mathrm{Arg}\,z$ (Unit A2, Section 5, Para. 1)

A streamline through i is given by $3\,\mathrm{Arg}\,z = \Psi(0,1) = 3\pi/2$. So Arg z = π/2.
The velocity function at i is q(i) = 3i (upwards)

A streamline through 1 - i is given by $3\,\mathrm{Arg}\,z = \Psi(1,-1) = -3\pi/4$. So Arg z = -π/4.
The velocity function at 1 - i is q(1 - i) = 3/(1+i) = 3(1 - i)/2 (South-east)



(c) 2 marks

Flux of q across the unit circle Γ = {z : |z| = 1} is (Unit D2, Section 2, Para. 1)

$$\mathrm{Im}\left(\int_\Gamma \overline{q}(z)dz\right) = \mathrm{Im}\left(\int_\Gamma \frac{3}{z}dz\right) = \mathrm{Im}(3 * 2\pi i) = 6\pi$$

by Cauchy's Integral Formula (Unit B2, Section 2, Para 1).

2000 Question 8

(a) 4 marks

If α is a fixed point of f then $f(\alpha) = \alpha$ (Unit D3, Section 1, Para. 3).

$f(\alpha) = \alpha \Leftrightarrow 2\alpha - 2i\alpha^2 = \alpha \Leftrightarrow \alpha(1 - 2i\alpha) = 0$.

Therefore the fixed points are at z = 0 and $z = \frac{1}{2i} = -\frac{1}{2}i$.

$f'(z) = 2 - 4iz$.

When z = 0 then $|f'(z)| = 2$. Therefore 0 is a repelling fixed point (Unit D3, Section 1, Para. 5).

When $z = -\frac{1}{2}i$. then $|f'(z)| = |2 + 2i^2| = 0$. Therefore $-\frac{1}{2}i$ is a super-attracting fixed point.

(b) 4 marks

(b)(i) $-1 + \frac{1}{5}i \in M$ (Unit D3 Section 4 Paras. 9(b) and 8).

(b)(ii) Let $c = \frac{1}{2} - i$.

$$P_c(0) = \frac{1}{2} - i.$$

$$P_c^2(0) = \left(\frac{1}{2} - i\right)^2 + \left(\frac{1}{2} - i\right) = \left(\frac{1}{4} - 1 - i\right) + \left(\frac{1}{2} - i\right) = -\frac{1}{4} - 2i.$$

As $|P_c^2(0)| > 2$ then c does not lie in the Mandelbrot set (Unit D3, Section 4, Para. 5).

2000 Question 9

(a) 7 marks

$f(z) = u(x, y) + i\, v(x, y)$
where $u(x,y) = x^2 + by^2$, and $v(x,y) = 2axy$.

$$\frac{\partial u}{\partial x}(x,y) = 2x, \qquad \frac{\partial u}{\partial y}(x,y) = 2by \qquad \frac{\partial v}{\partial x}(x,y) = 2ay, \qquad \frac{\partial v}{\partial y}(x,y) = 2ax$$

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous at each point of $\mathbb{C}$.

then, by the Cauchy-Riemann Converse Theorem (Unit A4, Section 2, Para. 3), f is differentiable at (α, β) if the Cauchy-Riemann equations (Unit A4, Section 2, Para. 1) are satisfied at that point.

$\frac{\partial u}{\partial x}(\alpha,\beta) = \frac{\partial v}{\partial y}(\alpha,\beta)$ when $2\alpha = 2a\alpha$. This is satisfied if $\alpha = 0$ or $a = 1$.

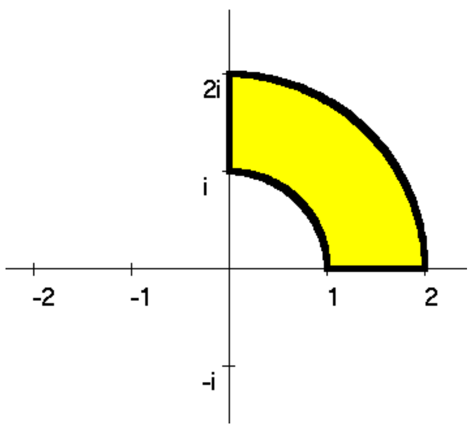
$\frac{\partial v}{\partial x}(\alpha,\beta) = -\frac{\partial u}{\partial y}(\alpha,\beta)$ when $2a\beta = -2b\beta$. This is satisfied if $\beta = 0$ or $a = -b$.

If f is analytic on $\mathbb{C}$ then the Cauchy-Riemann equations must hold everywhere in $\mathbb{C}$.
Therefore we must have $a = 1$ and $b = -a = -1$,

<< Note. When $a = 1$ and $b = -1$ then $f(z) = x^2 + 2ixy - y^2 = (x + iy)^2 = z^2$ >>

(b) 11 marks

(i)



(ii) $\gamma_3(t) = 2\exp(it)$ $\quad (t \in [0, \pi/2])$

$\gamma_4(t) = it \qquad (t \in [1, 2])$

(iii) (Unit A4, Section 4, Para. 3)

γ_1. Using the Restriction Rule (Unit A3, Section 2, Para. 7) the parametrization is differentiable on $[0, \pi/2]$ and $\gamma_1'(t) = i\exp(it)$. Since γ_1' is continuous and non-zero on $[0, \pi/2]$ the parametrization and the path are smooth.

γ_2. The parametrization is differentiable on $[1, 2]$ and $\gamma_2'(t) = 1$. Since γ_2' is continuous and non-zero on $[1, 2]$ the parametrization and the path are smooth.

γ_3. The parametrization is differentiable on $[0, \pi/2]$ and $\gamma_3'(t) = 2i\exp(it)$. Since γ_3' is continuous and non-zero on $[0, \pi/2]$ the parametrization and the path are smooth.

γ_4. The parametrization is differentiable on $[1, 2]$ and $\gamma_4'(t) = i$. Since γ_4' is continuous and non-zero on $[1, 2]$ the parametrization and path are smooth.

(ii) Log z is one of the standard functions (Unit A4, Section 3, Para. 4) and

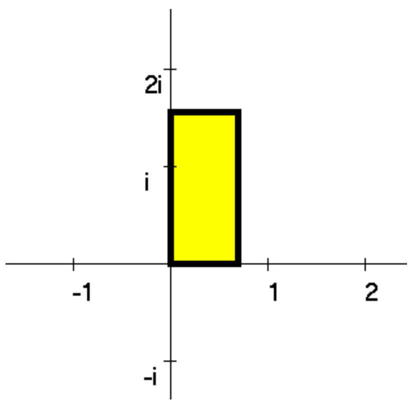
$$\text{Log}'(z) = 1/z$$

on the domain $\mathbb{C} - \{x \in \mathbb{R} : x \le 0\}$

Since Log z is analytic when $z \in \mathbb{C} - \{x \in \mathbb{R}: x \le 0\}$ and $\text{Log}'(z) \ne 0$ then Log is conformal on $\mathbb{C} - \{x \in \mathbb{R}: x \le 0\}$ (Unit A4, Section 4, Para. 6).

(iii) (unit A2, Section 5, Paras. 1 and 2)

$(\text{Log}\circ\gamma_1)(t) = it,$	$t \in [0, \pi/2].$
$(\text{Log}\circ\gamma_2)(t) = \text{Log}\, t = \log_e t,$	$t \in [1, 2].$
$(\text{Log}\circ\gamma_3)(t) = \text{Log}\, 2 + it = \log_e 2 + it,$	$t \in [0, \pi/2].$
$(\text{Log}\circ\gamma_4)(t) = \log_e \lvert it\rvert + i\,\text{Arg}(it) = \log_e t + i\pi/2,$	$t \in [1, 2].$



[$1 + i \in S$ and $\text{Log}(1 + i) = ½\log_e 2 + i\pi/4$ so S maps to the inside of the rectangle.
OR As we move from 1 to 2 on the original boundary S is on the left. Therefore as we move from $\log_e 1 = 0$ to $\log_e 2$ on the image of this boundary the image of S is also on the left]

2000 Question 10

(a) 10 marks

(a)(i) f has simple poles at z = 0 and z = 2.

(a)(ii) $f(z) = \dfrac{4}{z(z-2)} = -\dfrac{2}{z\left(1-\frac{z}{2}\right)}$

$$= -\frac{2}{z}\left\{\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n}\right\}$$

since |z / 2| < 1 on {z : 0 < |z| < 2} (Unit B3, Section 3, Para. 5)

Hence the required Laurent series about 0 is

$$-\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{n-1} = -\frac{2}{z} - 1 - \frac{z}{2} - \frac{z^2}{4} - \ldots - \left(\frac{z}{2}\right)^{n-1} - \ldots .$$

(a)(iii) $f(z) = \dfrac{4}{z(z-2)} = \dfrac{4}{(z-2)}\dfrac{1}{(z-2)+2} = \dfrac{4}{(z-2)^2}\dfrac{1}{1+\frac{2}{z-2}}$

$$= \frac{4}{(z-2)^2}\left\{\sum_{n=0}^{\infty}\left(\frac{-2}{z-2}\right)^{n}\right\}$$

since |2 / (z-2)| < 1 on {z : |z - 2| > 2} (Unit B3, Section 3, Para. 5)

Therefore the required Laurent series about 2 is

$$\sum_{n=0}^{\infty}\left(\frac{-2}{z-2}\right)^{n+2} = \frac{4}{(z-2)^2} - \frac{8}{(z-2)^3} + \frac{16}{(z-2)^4} - \ldots + \left(\frac{-2}{z-2}\right)^{n+2} - \ldots .$$

(b) Identical to 2004 Qu 10(b).

2000 Question 11

(a) 9 marks

(a)(i)

Putting $z = x + iy$ where $x, y \in \mathbb{R}$ then
$\exp(iz) = \exp(ix - y) = e^{-y}(\cos x + i \sin x)$

Since $| \exp z | = e^{\text{Re } z}$ (Unit A2, Section 4, Para. 2b) then
$| \exp(iz) | = \exp (e^{-y}\cos x)$

(a)(ii)

Let $f(z) = \exp(e^{iz})$ and $R = \{z : -\pi < \text{Re } z < \pi, -1 < \text{Im } z < 1\}$.

As f is analytic on the bounded region R and continuous on $\overline{R}$ then by the Maximum Principle (Unit C2, Section 4, Para. 4) there exists an $\alpha \in \partial R$ such that $| f(z) | \leq | f(\alpha) |$ for $z \in \overline{R}$.

From part (i) we have $| \exp(iz) | = \exp (e^{-y}\cos x)$.

As $e^{-y}\cos x$ is real we need to find the maximum of $e^{-y}\cos x$ on ∂R. e^{-y} is a maximum when $y = -1$ and $\cos x$ is a maximum when $x = 0$. These values can be attained simultaneously on ∂R.

Therefore $\max \{ \exp(e^{iz}) : -\pi \leq \text{Re } z \leq \pi, -1 \leq \text{Im } z \leq 1\} = e^e$.

The maximum only occurs when $z = -i$ as at all other points in $\overline{R}$ either
$e^y < e^1$ or $\cos x < 1$.

(b) 9 marks

Let $D_f = \{z: |z| < 3\}$ and $D_g = \{z: |z| > 3\}$.
Since $D_f \cap D_g = \varnothing$ then f and g are not direct analytic continuations of each other.

When $z \in D_f$ then $|z|/3 < 1$ and the geometric series $\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$ is convergent and has the sum

$$\frac{1}{1-\frac{z}{3}} = \frac{3}{3-z}.$$ (Unit B3, Section 3, Para. 5)

When $z \in D_g$ then $3/|z| < 1$ and the geometric series $\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n$ is convergent and has the sum

$$\frac{1}{1-\frac{3}{z}} = \frac{z}{z-3}.$$

Therefore $-\sum_{n=1}^{\infty}\left(\frac{3}{z}\right)^n = -\frac{3}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n = \frac{3}{3-z}$ when $z \in D_g$.

Let $h(z) = \frac{3}{3-z}$ on D_h, where $D_h = \mathbb{C} - \{3\}$.

Since $f = h$ when $z \in D_f \subseteq D_f \cap D_h$ then h is an analytic continuation of f.

Since $g = h$ when $z \in D_g \subseteq D_g \cap D_h$ then g is an analytic continuation of h.

Since (f, D_f), (g, D_g), (h, D_h) form a chain then f and g are indirect analytic continuations of each other.

2000 Question 12

Identical to 2004 Qu 12.