1999 Question 1

(a) 2 marks

$(1 + i)^4 = (1 + 2i - 1)^2 = (2i)^2 = -4$

(b) 3 marks

$$\cos(\pi - i\log_e 2) = \cos\pi\,\cos(i\log_e 2) + \sin\pi\,\sin(i\log_e 2)$$

(Unit A2, Section 4, Para. 5)

$$= -\cos(i\log_e 2)$$

$$= -\frac{\exp(-\log_e 2) + \exp(\log_e 2)}{2}$$

(Unit A2, Section 4, Para .4)

$$= -\frac{\frac{1}{2} + 2}{2} = -\frac{5}{4}$$

(c) 3 marks

$(-e)^{i\pi} = \exp(i\pi\,\mathrm{Log}(-e))$ (Unit A2, Section 5, Para. 3)

$= \exp(i\pi(\log_e|-e| + i\,\mathrm{Arg}(-e)))$ (Unit A2, Section 5, Para. 1)

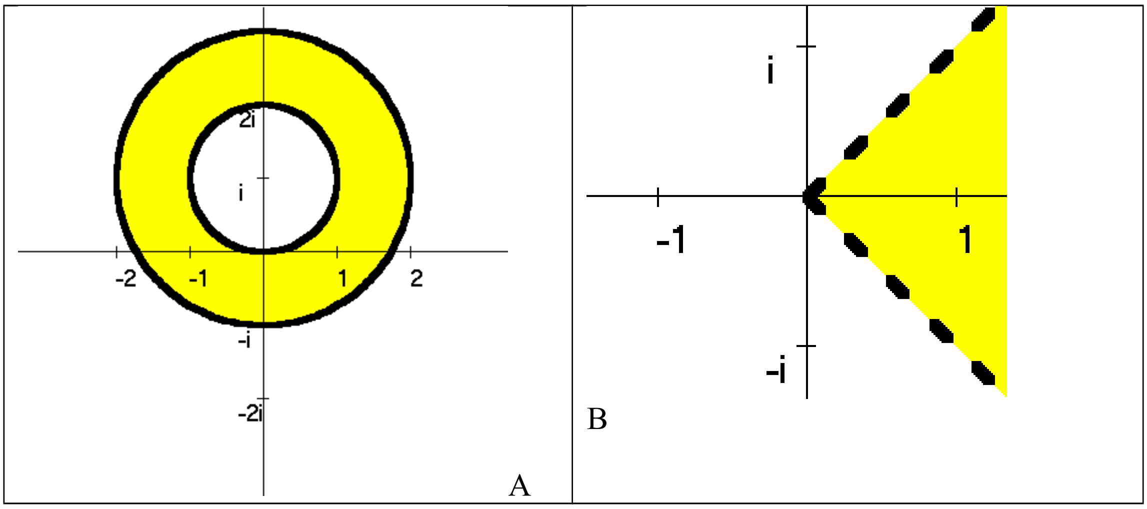
$= \exp(i\pi(1 + i\pi))$

$= \exp(i\pi)\exp(-\pi^2)$

$= -\exp(-\pi^2)$

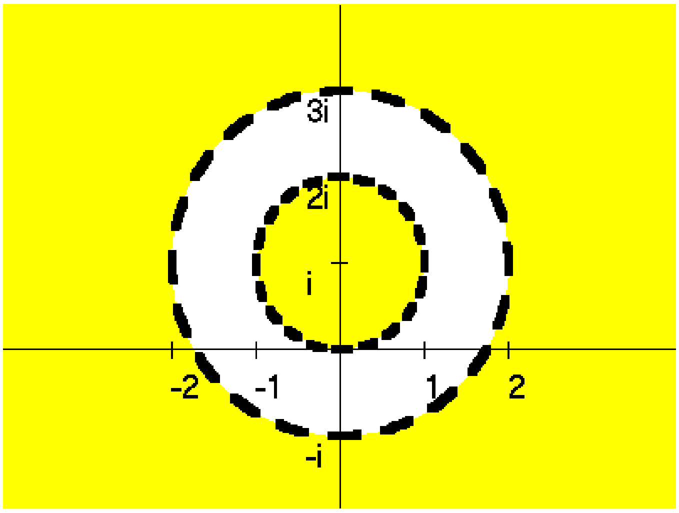
1999 Question 2

(a) 2 marks



(b) 6 marks

C = ext A (Unit A3, Section 5, Para. 9)



(b)(i) (Unit A3, Section 4, Paras. 6 and 7)

A is not a region since it is not open.
B is a region.
C is not a region as it is not connected.

(b)(ii) (Unit A3, Section 5, Para. 5)

A is compact.
B is not compact as it is not closed or bounded.
C is not compact as it is not closed or bounded.

1999 Question 3

(a) 4 marks (Unit B1, Ex. 2.1(ii) – Opposite direction along contour)

The standard parametrization (Unit A2, Section 2, Para. 3) for Γ is

$\gamma(t) = (1 - t)i + t, \quad t \in [0, 1]$

and $\gamma'(t) = 1 - i$.

As γ is differentiable on [0, 1], γ' is continuous on [0, 1], and γ' is non-zero on [0, 1] then γ is a smooth path (Unit A4, Section 4, Para. 3).

As γ is a smooth parametrization (Unit B1, Section 2, Para. 1) then

$$\int_{\Gamma_1} \operatorname{Im} z \, dz = \int_0^1 \{ \operatorname{Im} \gamma(t) \} \gamma'(t) \, dt$$
$$= \int_0^1 (1 - t)(1 - i) \, dt$$
$$= (1 - i)\left[- \frac{(1 - t)^2}{2} \right]_0^1$$
$$= \frac{1 - i}{2}$$

(b) 4 marks

The length of C is L = $2\pi * 2 = 4\pi$.

Using the Triangle Inequality (Unit A1, Section 5, Para. 3b) then, for $z \in C$, we have

$$|\bar{z}^2 - 1| \le |\bar{z}^2| + 1 = |z|^2 + 1 = 4 + 1 = 5$$

Using the Backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 2b) then, for $z \in C$, we have

$$|z^2 - 1| \ge \left| |z^2| - 1 \right| = |4 - 1| = 3$$

Putting $f(z) = \dfrac{\bar{z}^2 - 1}{z^2 - 1}$ we have $|f(z)| \le 5/3 = M$ for $z \in C$.

By the Quotient Rule (Unit A3, Section 2, Para. 5) f(z) is continuous on $\mathbb{C} - \{-1, 1\}$ and hence on the circle C.

Therefore by the Estimation Theorem (Unit B1, Section 4, Para. 3)

$$\left| \int_\Gamma f(z) \, dz \right| \le ML = \frac{5}{3} * 4\pi = \frac{20}{3}\pi$$

1999 Question 4

(a) 6 marks

(a)(i) $\mathbb{C}$ is a simply-connected region (Unit B2, Section 1, Para. 3), C is a simple-closed contour (Unit B2, Section 1, Para. 1) in $\mathbb{C}$, and f(z) = exp (iπz) is analytic on $\mathbb{C}$.

As -1 lies inside the circle C then by Cauchy's Integral formula (Unit B2, Section 2, Para. 1)

$$\int_C \frac{e^{i\pi z}}{z+1}\,dz = 2\pi i\, f(-1) = 2\pi i * e^{-i\pi} = -2\pi i$$

(a)(ii) Let $\boldsymbol{R} = \{z \in \mathbb{C} : |z - i| < 5^{1/2}\}$. $\boldsymbol{R}$ is a simply-connected region (Unit B2, Section 1, Para. 3) and C is a simple-closed contour in $\boldsymbol{R}$. As $\dfrac{e^{i\pi z}}{z+2}$ is analytic on $\boldsymbol{R}$ then by Cauchy's Theorem (Unit B2, Section 1, Para. 4)

$$\int_C \frac{e^{i\pi z}}{z+2}\,dz = 0$$

(b) 2 marks

Since $1 + z^2$ and cos are entire functions then by the Composition rule (Unit A4, Section 3, Para. 1) so is $f(z) = \cos(1 + z^2)$.

By Liouville's theorem (Unit B2, Section 2, Para. 2) if f is a bounded entire function then f is constant.

Since $f(i) = \cos 0 = 1 \neq \cos 1 = f(0)$ then f is not constant. Therefore f is not bounded on $\mathbb{C}$ so there is a complex number z such that $|\cos(1 + z^2)| > 100$.

1999 Question 5

Same as 2004 Qu 5 apart from 4 marks being awarded for each part.

1999 Question 6

Same as 2004 Qu 6.

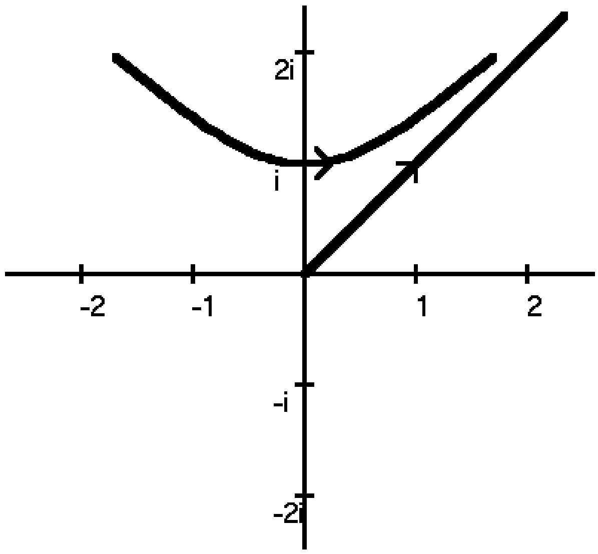
1999 Question 7

Equivalent to 2004 Qu 7 except in part (b) also need to calculate the stream line through i.

A streamline through i is given by $\frac{1}{2}\left(-x^2+y^2\right)=\Psi(0,1)=\frac{1}{2}$.

Therefore the streamline has the equation $y^2 = x^2 + 1$.

At i the velocity function q(i) = i (- i) = 1 (positive x direction)



1999 Question 8

(a) 3 marks

Since $f(i) = i^2 - i^2 + i = i$ then i satisfies the fixed point equation (Unit D3, Section 1, Para. 3). Therefore i is a fixed point of f.

$$f'(z) = 2z - i.$$

As $|f'(i)| = |i| = 1$ then i is an indifferent fixed point of f. (Unit D3, Section 1, Para. 5)

(b) 5 marks

(b)(i) [Unit D3, Problem 4.3(b)]

Let $c = 1 + i$.
b
$P_c(0) = 1 + i$.
$P_c^2(0) = (1+i)^2 + (1+i) = 2i + (1+i) = 1 + 3i$.

As $|P_c^2(0)| > 2$ then c does not lie in the Mandelbrot set (Unit D3, Section 4, Para. 5).

(b)(ii) Let $c = -\frac{9}{10} - \frac{\sqrt{3}}{10}i$.

Since $|c+1| = \left|\frac{1}{10} - \frac{\sqrt{3}}{10}i\right| = \frac{1}{5} < \frac{1}{4}$ then P_c has an attracting 2-cycle (Unit D3, Section 4, Para. 9). Therefore c belongs to the Mandelbrot set (Unit D3, Section 4, Para. 8).

1999 Question 9

(a) 8 marks

(a)(i)

$f(z) = z + |z|^2 = (x + iy) + (x^2 + y^2) = u(x, y) + iv(x, y)$,
where $u(x,y) = x + x^2 + y^2$, and $v(x,y) = y$.

$$\frac{\partial u}{\partial x}(x, y) = 1 + 2x,\ \frac{\partial u}{\partial y}(x, y) = 2y,\ \frac{\partial v}{\partial x}(x, y) = 0,\ \frac{\partial v}{\partial y}(x, y) = 1$$

(a)(ii)

If f is differentiable then the Cauchy-Riemann equations hold (Unit A4, Section 2, Para. 1). If they hold at (a, b)

$$\frac{\partial u}{\partial x}(a, b) = 1 + 2a = 1 = \frac{\partial v}{\partial y}(a, b), \text{ and}$$
$$\frac{\partial v}{\partial x}(a, b) = 0 = -2b = -\frac{\partial u}{\partial y}(a, b)$$

Therefore the Cauchy-Riemann equations only hold at (0, 0).

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous at (0, 0).
3. satisfy the Cauchy-Riemann equations at (0, 0)

then, by the Cauchy-Riemann Converse Theorem (Unit A4, Section 2, Para. 3), f is differentiable at 0.

As the Cauchy-Riemann only hold at (0, 0) then f is not differentiable on any region surrounding 0. Therefore f is not analytic at 0. (Unit A4, Section 1, Para. 3)

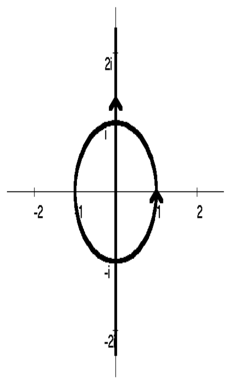
(a)(iii)

$f'(0, 0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 1$ (Unit A4, Section 2, Para. 3).

(b) 10 marks

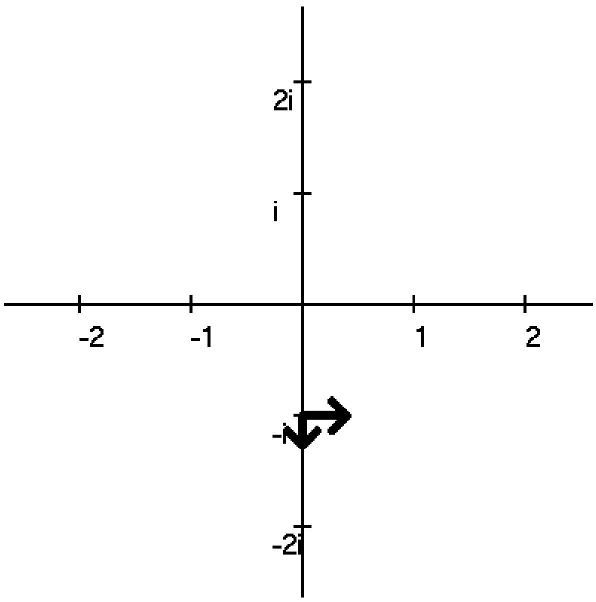
(i) The domain of g is $\mathbb{C}$ (Unit A4, Section 1, Para. 7) and its derivative $g'(z)=3z^2$ also has domain $\mathbb{C}$ (Unit A4, Section 3, Para. 4) . Therefore g is analytic at i and since $g'(i) = -3 \neq 0$ then g is conformal at i (Unit A4, Section 4, Para. 6).

(ii) $\pi/2$ is in the domain of γ_1 so $\gamma_1(\pi/2) = e^{i\pi/2} = i$.
1 belongs to the domain of γ_2 so $\gamma_2(1) = i$. Therefore Γ_1 and Γ_2 meet at the point i.



(iii) As g is analytic on $\mathbb{C}$ and $g'(i) \neq 0$ then a small disc centred at i is mapped approximately (Unit A4, Section 1, Para. 11) to a small disc centred at $g(i) = -i$. The disc is rotated by Arg $(g'(i))$ = Arg $-3 = \pi$, and scaled by a factor $|g'(i)| = 3$.

The horizontal line in the diagram below is $g(\Gamma_1)$. (Unit A4, Section 4, Para. 4)



1999 Question 10

(a) 7 marks

$\sin z = z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right)$, for $z \in \mathbf{C}$. (Unit B3, Section 3, Para. 5)

$$\frac{z}{\sin z} = \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right)^{-1}$$
$$= 1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \ldots\right)^2 + \ldots$$
$$= 1 + \frac{z^2}{6} + z^4\left(-\frac{1}{120} + \frac{1}{36}\right) + \ldots$$

Therefore the Laurent series about 0 for f is $1 + \frac{z^2}{6} + \frac{7}{360}z^4 + \ldots$ for $0 < |z| < \pi$

$\frac{1}{z^2 \sin z} = \frac{f(z)}{z^3}$ is analytic on the punctured disc $\mathbb{C} - \{0\}$. It has the Laurent series about 0

$$\frac{1}{z^3} + \frac{1}{6z} + \frac{7}{360}z + \ldots = \sum_{n=-\infty}^{\infty} a_n z^n$$

As C is a circle with centre 0 then (Unit B4, Section 4, Para. 2)

$$\int_C \frac{f(z)}{z^3}\,dz = 2\pi i a_{-1} = \frac{\pi i}{3}.$$

(b) 5 marks

The domain of $A = \mathbb{C} - \{n\pi : n \in \mathbb{Z}\}$.

Suppose that g is another analytic function with domain A which agrees with f on $\{iy : y > 0\}$

The set $S = \left\{i\left(1 + \frac{1}{n}\right) : n = 1,2,3,\ldots\right\} \subseteq A$ and has the limit point $i \in A$.

f agrees with g throughout the set $S \subseteq A$ and S has a limit point which is in A. Therefore by the Uniqueness theorem (Unit B3, Section 5, Para. 7) f agrees with g throughout A. Hence f is the only analytic function with domain A such that $f(iy) = \frac{y}{\sinh y}$ for $y > 0$.

(c) 6 marks

Since sin z = 0 when z = 0, z = ±π, z = ±2π, …. Then f(z) has singularities of the form kπ, k ∈ ℤ.

Singularity at z = 0.

At z = 0 we can use the Laurent series found in part (a). Since f(0) = 1 then the singularity at 0 is a removable singularity.

Singularities at z = kπ where k ∈ ℤ – {0}.

sin (z - kπ) = sin z * cos kπ - cos z * sin kπ = $(-1)^k$ sin z.

Therefore $f(z) = \frac{z}{\sin z} = (-1)^k \frac{z}{\sin(z-k\pi)} = (-1)^k \left\{ \frac{z-k\pi}{\sin(z-k\pi)} + \frac{k\pi}{\sin(z-k\pi)} \right\}$

$$\lim_{z \to k\pi} \frac{z-k\pi}{\sin(z-k\pi)} = 1$$

As $\lim_{z \to k\pi} (z-k\pi)\frac{k\pi}{\sin(z-k\pi)} = k\pi$ then there is a simple pole at z = kπ (Unit B4, Section 3, Para. 2).

Therefore there f has simple poles at kπ where k ∈ ℤ – {0}.

1999 Question 11

(a) 6 marks

Since $f(z) = \dfrac{\pi \cot \pi z}{9(z - i/3)(z + i/3)}$ then f has simple poles at z = ±i/3.

By the cover-up rule (Unit C1, Section 1, Para. 3)

$$\text{Res}(f, i/3) = \frac{\pi \cot(i\pi/3)}{9(i/3 + i/3)} = \frac{\pi \cot(i\pi/3)}{6i}, \text{ and}$$

$$\text{Res}(f, -i/3) = \frac{\pi \cot(-i\pi/3)}{9(-i/3 - i/3)} = \frac{\pi \cot(-i\pi/3)}{-6i}.$$

Since sin(iz) = i sinh z and cos(iz) = cosh z then cot(iz) = - i coth(z).

Therefore $\text{Res}(f, i/3) = -\dfrac{\pi \coth(\pi/3)}{6}$ and

$$\text{Res}(f, -i/3) = \frac{\pi \coth(-\pi/3)}{6} = -\frac{\pi \coth(\pi/3)}{6}.$$ (Unit A2, Section 4, Para. 6)

f(z) = g(z) / h(z) where $g(z) = \dfrac{\pi \cos \pi z}{9z^2 + 1}$ and h(z) = sin πz.

g and h are analytic at 0, h(0) = 0, and $h'(0) = \pi \cos(0) = \pi \neq 0$.

Therefore by the g/h rule (Unit C1, Section 1, Para. 2)

$$\text{Res}(f,0) = \frac{g(0)}{h'(0)} = \frac{\pi * 1}{1} * \frac{1}{\pi} = 1.$$

[You could also use Unit C1, Section 4, Para 1 – last line]

(b) 8 marks

The method given in Unit C1, Section 4, Para. 1 will be used.

$f(z) = \pi \cot \pi z * \phi(z)$ where $\phi(z) = 1/(9z^2 + 1)$.

ϕ is an even function which is analytic on $\mathbb{C}$ except for simple poles at the non-integral points $z = \pm i/3$.

Let S_N be the square contour with vertices at $(N + ½)(\pm 1 \pm i)$.

On S_N we have $|z| \geq N + ½$ so, using the backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 2),

$$|9z^2 + 1| \geq |\,|9z^2| - 1\,| \geq 9(N + ½)^2 - 1 \geq 9N^2.$$

On S_N we also have $\cot \pi z \leq 2$ (Unit C1, Section 4, Para. 2) so on C_N

$$|\,f(z)\,| \leq \frac{\pi(2)}{9N^2}.$$

The length of the contour S_N is $4(2N + 1)$.

As f is continuous on the contour S_N then by the Estimation Theorem (Unit B1, Section 4, Para. 3) we have

$$\left|\int_{S_N} f(z)dz\right| \leq \frac{2\pi}{9N^2} 4(2N+1) = \frac{8\pi}{9N}\left(2 + \tfrac{1}{N}\right).$$

Hence $\lim_{N \to \infty} \left|\int_{S_N} f(z)dz\right| = 0$.

Therefore the conditions specified in Unit C1, Section 4, Para. 1 hold so

$$\sum_{n=1}^{\infty} \frac{1}{9n^2+1} = -\frac{1}{2}\left(\text{Res}(f,0) + \text{Res}(f,i/3) + \text{Res}(f,-i/3)\right)$$
$$= -\frac{1}{2} + \frac{\pi}{6}\coth\frac{\pi}{3}.$$

(c) 4 marks

$$\sum_{n=-\infty}^{\infty} \frac{1}{9n^2+1} = \sum_{n=-\infty}^{-1} \frac{1}{9n^2+1} + 1 + \sum_{n=1}^{\infty} \frac{1}{9n^2+1}$$
$$= 1 + 2\sum_{n=1}^{\infty} \frac{1}{9n^2+1} = \frac{\pi}{3}\coth\frac{\pi}{3}.$$

1999 Question 12

(a) 3 marks

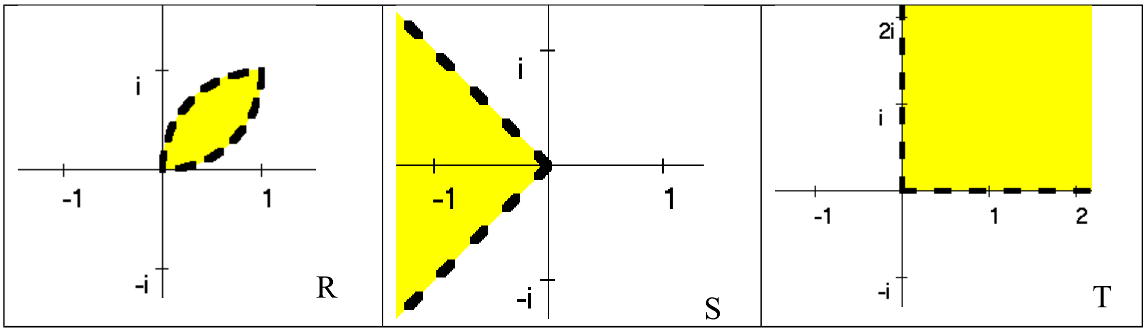
Using the formula for a transformation mapping points to the standard triple (Unit D1, Section 2, Para. 11) then $\hat{f}_1$ corresponds to

$$f_1(z) = \frac{(z-0)}{(z-(1+i))}\frac{(\infty-(1+i))}{(\infty-0)} = \frac{z}{z-(1+i)}$$

So $$\hat{f}_1\left(\tfrac{1}{2}(1+i)\right) = \frac{\tfrac{1}{2}(1+i)}{\tfrac{1}{2}(1+i)-(1+i)} = -1$$

(b) 15 marks

(b)(i)



(b)(ii)

As f_1 is a M✱bius transformation and R is a region in the domain of f_1 then $f_1(R)$ is a region and the boundary of R maps onto the boundary of f(R) (Unit D1, Section 4, Para. 3).

The boundary of R is $B_1 \cup B_2$ where

$B_1 = \{z : |z-1| = 1, \pi/2 \le \text{Arg}(z-1) \le \pi\}$ and

$B_2 = \{z : |z-i| = 1, -\pi/2 \le \text{Arg}(z-i) \le 0\}$.

As $\hat{f}_1$ maps 0 to 0 and 1 + i to ∞ then both B_1 and B_2 are mapped to extended lines which start at the origin. Since the angle between B_1 and B_2 at 0 is $\pi/2$ and the transformation $\hat{f}_1$ is conformal, then the angle between the extended lines at the origin is also $\pi/2$.

Since the angle between the boundaries of R at 0 and a line joining the origin to

½ (1 + i) is π/4 then as $\hat{f}_1$ is conformal then this is also true in $\hat{f}_1$ (R). From part (a) we have $\hat{f}_1\left(\frac{1}{2}(1+i)\right) = -1$ so the 2 boundaries must be mapped to extended lines at angles π/4 above and below the negative real-axis.

Therefore $\hat{f}_1(R) = S$.

(b)(iii)

w = g(z) = z exp(-3πi /4) maps S to T.

Therefore the extended conformal mapping from R to T is $\hat{f}$, where

$$f(z) = (g \circ f_1)(z) = \frac{z\exp(-3\pi i/4)}{z-(1+i)} = \frac{-z\frac{1}{\sqrt{2}}(1+i)}{z-(1+i)}$$

$$= \frac{-\sqrt{2}z}{(1-i)z-2}$$

(b)(iv)

Let $f(z) = \frac{-\sqrt{2}z}{(1-i)z-2} = \frac{az+b}{cz+d}$

Using the formula for the inverse function (Unit D1, Section 2, Para. 6) we have

$$f^{-1}(z) = \frac{-2z}{-(1-i)z-\sqrt{2}} = \frac{2z}{(1-i)z+\sqrt{2}}$$